COMBINATORICA
Akadémiai Kiadó – Springer-Verlag

COMBINATORICA 14 (3) (1994) 301–306

EVERY ARRANGEMENT EXTENDS TO A SPREAD

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Received December 5, 1991
Revised July 15, 1992

An *arrangement of pseudolines* in the Euclidean plane $\mathbf{E}^2$ is a finite family of simple curves, each asymptotic to some line at both "ends", every two of which intersect at precisely one point, at which they cross. Such arrangements, which exhibit many of the properties of straight line arrangements, have been studied since the work of F. Levi [2]; see [1] for an extensive bibliography up to 1971.

A *spread of pseudolines* in $\mathbf{E}^2$ is the continuous version of an arrangement: an infinite family of simple curves, each asymptotic to some line at both "ends", such that:

1. every two curves intersect at precisely one point, at which they cross;
2. there is a bijection L from the unit circle C to the family of curves such that $L(p)$ is a continuous function of $p \in C$.

Motivated by the fact that any finite arrangement of straight lines can be extended to a spread of straight lines, B. Grünbaum conjectured in [1] that the same should hold for pseudolines. It is this conjecture that we establish below:

Theorem. *Every arrangement of pseudolines in $\mathbf{E}^2$ may be extended to a spread of pseudolines.*

Proof. We begin by mapping $\mathbf{E}^2$ to the interior of a disk, in such a way that pseudolines in $\mathbf{E}^2$ map to curves on the disk with endpoints on the circle bounding the disk. Moreover it is easily seen by induction that an arrangement of pseudolines in a disk is combinatorially equivalent to an arrangement of (piecewise linear)

* Supported in part by NSA grant MDA904–89–H–2038, PSC-CUNY grant 662330, and the Center for Discrete Mathematics and Theoretical Computer Science (DIMACS), a National Science Foundation Science and Technology Center under NSF grant STC88–09648.

† Supported in part by NSF grant CCR–8901484, NSA grant MDA904–89–H–2030, and DIMACS.

‡ Supported in part by DIMACS.

& Supported in part by DIMACS.

AMS subject classification code (1991): 51 H 10, 51 A 45; 05 B 99, 51 D 20, 52 A 10, 52 C 99, 57 N 05

pseudolines in a regular $2n$-gon such that each face in the arrangement is a convex polygon, the pseudolines having antipodal vertices on the $2n$-gon as endpoints. Hence to prove the theorem we need only show that such a finite family of polygonal pseudolines in the $2n$-gon can be extended to an infinite family in which every point p on the boundary lies on exactly one pseudoline $L(p)$, with $L(p)$ a continuous function of p.

Let l and l' be two piecewise linear curves in the $2n$-gon P with distinct antipodal endpoints p, $\overline{p}$ and p', $\overline{p}'$, respectively, and let us drop for a moment the restriction that they meet (and cross) at precisely one point. Let q be an isolated point of intersection of l and l' at which they cross. Thus some small topological disk Δ contains q and no other point of intersection of l and l', and l and l' intersect $\partial\Delta$ at four points, s, s', $\overline{s}$, and $\overline{s}'$, lying between q and p, p', $\overline{p}$, and $\overline{p}'$, respectively. (See Figure 1.) We say that q is a *proper intersection point of* l and l' if l and l' cross at q and if s, s', $\overline{s}$, $\overline{s}'$ occur in the same order around Δ (clockwise or counterclockwise) as p, p', $\overline{p}$, $\overline{p}'$ do around P.

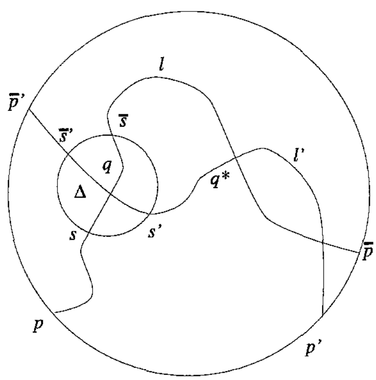

Fig. 1. q is a proper intersection point, Q is not*

We can then replace the global condition that curves intersect at precisely one point, at which they cross, by the local condition that every point of intersection is proper.

Lemma 1. *Two piecewise linear curves with antipodal endpoints on a disk intersect at precisely one point, at which they cross, if and only if every point of intersection of the two curves is a proper intersection point.*

Proof. Since the endpoint of each curve are antipodal, the first curve separates the endpoints of the second and thus the curves must have at least one intersection point. If there is only one, that intersection must clearly be proper. One the other hand, if two piecewise linear curves l and l' intersect at more than one point, we can list the points of intersection in order along l. Let q and q^* be two successive points of intersection. It is not hard to see that if q is proper then q^* is not. ∎

It is therefore sufficient to extend our arrangement of pseudolines to a continuous family such that every intersection is proper. Let $\mathscr{L}$ be our arrangement of n pseudolines in the regular $2n$-gon P, with endpoints at antipodal vertices, and with each polygonal pseudoline having a vertex only at the intersection of two or more pseudolines. The members of $\mathscr{L}$ partition P into a 2-dimensional cell complex $C(\mathscr{L})$, consisting of a set of (open) faces $F(\mathscr{L})$, (relatively open) edges $E(\mathscr{L})$, and vertices $V(\mathscr{L})$. Edges of P are considered as belonging to $E(\mathscr{L})$, and their endpoints as belonging to $V(\mathscr{L})$.

Let $\mathscr{A}$ be the set of (open) edges of the polygon P. For each $a \in \mathscr{A}$, there are two pseudolines, l_a, $l'_a \in \mathscr{L}$, beginning at the endpoints of a. Let $\gamma_a = l_a \cap l'_a$. Each edge $a \in \mathscr{A}$ also has an antipodal edge $\overline{a} \in \mathscr{A}$, and we have $\{l_a, l'_a\} = \{l_{\overline{a}}, l'_{\overline{a}}\}$.

Fix an edge $a \in \mathscr{A}$. We define an order relation $R_a(\mathscr{L})$ on the cell complex $\mathscr{L}$, as follows. Assume edge e and vertex v lie in the interior of polygon P and on the boundary ∂f of face f. Let $e \prec_a f$ (or simply $e \prec f$, if the subscript is clear from context) if the pseudoline $l \in \mathscr{L}$ containing e separates a from f; otherwise, let $e \succ_a f$. Let $v \prec_a f$ if *all* the pseudolines $l \in \mathscr{L}$ containing v separate a from f; let $v \succ_a f$ if all the pseudolines $l \in \mathscr{L}$ containing v separate $\overline{a}$ from f. Finally, for each $a \in \mathscr{A}$ on the boundary of a face f, let $a \prec_a f$ and $a \succ_{\overline{a}} f$.

For a given edge $e \in E(\mathscr{L}) \setminus \mathscr{A}$ we then have $e \prec_a f$ and $e \succ_a f'$ for exactly one f and one f'. Similarly, for a given vertex v in the interior of P we have $v \prec_a f$ and $v \succ_a f'$ for exactly one f and one f'. (See Figure 2.)

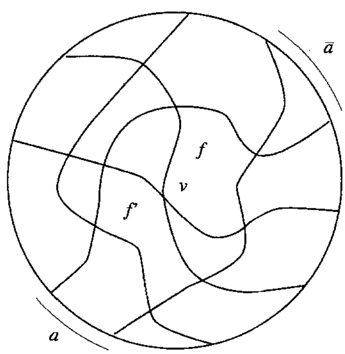

Fig. 2. Order relation $f \prec_a v \prec_a f$

Lemma 2. *$R_a(\mathscr{L})$ extends to a partial order on the cell complex $C(\mathscr{L})$.*

Proof. Suppose there were a cycle

$$x_0 \prec x_1 \prec \ldots \prec x_k = x_0$$

in $R_a(\mathscr{L})$. Without loss of generality, we can assume that x_0 is a face of the arrangement $\mathscr{L}$. Then some pseudoline l containing x_1 must strictly separate x_0 from $\overline{a}$. On the other hand, l must also strictly separate every x_i, $i > 1$, from a.

Since $x_k = x_0$ cannot be strictly separated from both a and $\overline{a}$ by l, there can be no cycle in $R_a(\mathscr{L})$. ∎

Let $R_a^*(\mathscr{L})$ be the transitive closure of $R_a(\mathscr{L})$, and let us write $\prec_a$ (or simply $\prec$) for this more general relation as well. (Intuitively, two cells c_1 and c_2 stand in the relation $c_1 \prec_a c_2$ whenever c_1 is "closer to a" than c_2 is.) Let G_a be the set of all faces, edges, and vertices that are related to γ_a:

$$G_a = \{g \in F(\mathscr{L}) \cup E(\mathscr{L}) \cup V(\mathscr{L}) : g \prec_a \gamma_a \text{ or } g \succ_a \gamma_a \text{ or } g = \gamma_a\}.$$

The pseudolines l_a and l'_a partition P into four (open) regions. Thus G_a consists of γ_a, a and $\overline{a}$, together with all the faces, edges, and vertices lying in the two open regions adjacent to a and $\overline{a}$.

For each $a \in \mathscr{A}$ and $f \in G_a$, we define the "bottom" of f with respect to a by:

$$B_a(f) = \{p \in \partial f : p \in g \in G_a,\ g \prec_a f\}.$$

(Notice that if f is the face having a on its boundary, then $B_a(f) = a$.)

If $g \in G_a$ is an edge or a vertex, $g \neq \overline{a}$, then there is a unique minimal face f such that $g \prec_a f$. Thus for $p \in g$ there is a unique face f such that $p \in B_a(f)$. In particular, let f_a^* be the unique face for which $\gamma_a \in B(f_a^*)$; notice that $B_a(f_a^*) = \gamma_a$.

Lemma 3. *For every face $f \in G_a \setminus \{f_a^*\}$, $B_a(f)$ is an open, non-empty arc.*

Proof. Let $e_1, \ldots, e_k$ be the edges of f in cyclic order, and $l_1, \ldots, l_k$ the pseudolines containing them. Direct each l_i so that as ∂f is traced in the counterclockwise direction, each edge e_i is traversed positively. Since l_i and l_{i+1} (the numbering being modulo k) meet at a vertex of f, they meet nowhere else; hence their initial points, as well as their terminal points, must fall on P in the same cyclic order. It follows that those edges whose extensions separate a from f form just a single connected sequence, which is necessarily non-empty if $f \neq f_a^*$, from which the conclusion follows. ∎

Let $B_a^-(f)$ and $B_a^+(f)$ be the endpoints of $B_a(f)$, with $B_a^-(f)$, $B_a(f)$, and $B_a^+(f)$ occuring in counterclockwise order around f.

Lemma 4. *If a, b, $\overline{a}$, $\overline{b} \in \mathscr{A}$ occur in counterclockwise order around P, and if $f \in G_a \cap G_b$, then $B_a^-(f)$, $B_b^-(f)$, $B_{\overline{a}}^-(f)$, $B_{\overline{b}}^-(f)$ occur in (not necessarily strictly) counterclockwise order around f; likewise for $B_a^+(f)$, $B_b^+(f)$, $B_{\overline{a}}^+(f)$, $B_{\overline{b}}^+(f)$.*

Proof. It follows by the same observation used in proving Lemma 3 that replacing a by $\overline{a}$, for example, produces a cyclic shift in the edges of f that contribute to $B_a(f)$.

Lemma 5. *There exists a family of mappings*

$$(\psi_a : B_a(f) \to B_{\overline{a}}(f))_{a \in \mathscr{A}, f \in G_a \setminus (f_a^* \cup f_{\overline{a}}^*)}$$

such that:

1. *each ψ_a is a homeomorphism;*
2. *$\psi_{\overline{a}}$ is the inverse of ψ_a for each a;*

3. *for every distinct pair of points p, $p' \in B_a(f)$, the line segment joining p to $\psi_a(p)$ does not cross the line segment joining p' to $\psi_a(p')$;*
4. *if $p \in B_a(f) \cap B_b(f)$ and a, b, $\overline{a}$, $\overline{b}$ occur in counterclockwise order around P, then p, $\psi_a(p)$, $\psi_b(p)$ are distinct points occuring in counterclockwise order around f.*

Proof. Fix a face $f \in G_a \backslash (f_a^* \cup f_{\overline{a}}^*)$ for some $a \in \mathscr{A}$, and consider its boundary as a (topological) circle. For convenience, let $s(a) = B_a(f)$ for each $a \in \mathscr{A}$. We define the maps ψ_a, $\psi_{\overline{a}}$ for successive pairs a_1, $\overline{a}_1$; a_2, $\overline{a}_2$; ... as follows: Choose a pair a, $\overline{a}$ and let ψ_a be an arbitrary circular-order-reversing homeomorphism from $s(a)$ onto $s(\overline{a})$, and $\psi_{\overline{a}}$ its inverse. Now suppose $\psi_{a'}$ and $\psi_{\overline{a}'}$ have been defined for $a' \in \mathscr{A}' \subset \mathscr{A}$ so as to satisfy the desired condition, and suppose a, $\overline{a} \in \mathscr{A} \backslash \mathscr{A}'$. Let $\mathscr{A}'' = \{a' \in \mathscr{A}' : f \in G_{a'}\}$. Let a_- and a_+ be the members of $\mathscr{A}''$ immediately preceding and following a in counterclockwise order. If $s(a_-)$ or $s(a_+)$ does not overlap $s(a)$, we need not worry about it when defining ψ_a. Suppose one of these arcs, say $s(a_-)$, does overlap $s(a)$. Then when defining ψ_a we need only take care that for $p \in s(a_-) \cap s(a)$ the points p, $\psi_{a_-}(p)$, $\psi_a(p)$ are distinct points lying in counterclockwise order around f. But this can be done, precisely because of Lemma 4.

If we construct these maps around the boundary of each face f, and then join corresponding points by straight lines, it is immediate that condition (3) is satisfied. ∎

Now extend each function ψ_a to $B_a(f_a^*)$, by letting $\psi_a(p) = \gamma_a$ for $p \in B_a(f_a^*)$. Notice that for $p \in a$, $\psi_a^k(p) = \gamma_a$ for some $k \geq 0$; this follows from Lemma 3 and the finiteness of the situation.

Let

$$(p, \psi_a(p), \psi_a^2(p), \ldots, \psi_a^k(p) = \gamma_a)$$

be the polygonal curve consisting of all the line segments $(\psi_a^i(p), \psi_a^{i+1}(p))$, $0 \leq i < k$. For each $p \in a \in \mathscr{A}$ there is an antipodal point $\overline{p} \in \overline{a} \in \mathscr{A}$. Let $L(p)$ be the union of the polygonal curves

$$(p, \psi_a(p), \psi_a^2(p), \ldots, \psi_a^k(p) = \gamma_a) \text{ and } (\overline{p}, \psi_{\overline{a}}, \psi_{\overline{a}}^2(\overline{p}), \ldots, \psi_{\overline{a}}^{k'}(\overline{p}) = \gamma_a).$$

To complete the proof of the theorem, we will show that

$$\mathscr{S} = \{L(p) : p \in a \in \mathscr{A}\} \cup \mathscr{L}$$

is a spread of pseudolines.

Let l and l' be two members of $\mathscr{S}$. If $l \in \mathscr{L}$ and $l' \in \mathscr{L}$, then, by definition, they intersect in exactly one point.

Suppose $l \in \mathscr{L}$ but $l' \notin \mathscr{L}$; say $l' = L(p)$ for $p \in a \in \mathscr{A}$. Since the endpoints of l and of l' are each antipodal, l and l' must intersect in at least one point $q^* \in g^* \in G_a$, with g^* either an edge or a vertex belonging to l. By the construction of l', if l' intersects each of g, $g' \in G_a$ then either $g \prec_a g'$ or $g \succ_a g'$. On the other hand, no two vertices or edges belonging to l are comparable under $R_a^*(\mathscr{L})$, by Lemma 2. Thus l' can meet l only once.

Suppose $l \notin \mathscr{L}$ and $l' \notin \mathscr{L}$. If l and l' have endpoints in the same arc $a \in \mathscr{A}$, then by construction l and l' intersect only at γ_a, where they cross. Suppose l and l' have endpoints in different arcs, a and a', respectively. Using the properties of ψ_a

and $\psi_{a'}$ given in Lemma 5, it follows that every intersection of l and l' is proper. By Lemma 1, it then follows that l and l' intersect precisely once.

For each point p on the boundary of P, there is a unique pseudoline $L(p)$ starting at p. Since ψ_a is a continuously varying function of $p \in a$, and $L(p)$ is determined by iterating ψ_a, $L(p)$ also varies continuously with $p \in a$. Finally, the continuity of L at the endpoints of each arc a follows from the surjectivity (and monotonicity) of ψ_a. Thus $\mathscr{S}$ is a spread of pseudolines containing $\mathscr{L}$. ∎

[*Added in June, 1992:* Using a different method, the authors have recently succeeded in proving a stronger conjecture of Grünbaum's, to the effect that every pseudoline arrangement extends to a topological plane, and in fact that there is a topological plane containing an isomorphic copy of *every* pseudoline arrangement; details to appear in the American Mathematical Monthly.]

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