Arrangements and Topological Planes

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1. INTRODUCTION. Let Γ be a finite family of simple curves in the plane. When is there a homeomorphism of the plane to itself that takes all the curves in Γ to straight lines?

In the Euclidean plane, $\mathbf{E}^2$, we are faced with the fact that two non-intersecting curves in our family must map to two parallel lines. This introduces extraneous technical complications that only distract from the essence of the problem. As with many other geometric questions, it is much simpler to avoid the special cases caused by parallel lines by moving to the projective plane. The real projective plane $\mathbf{P}^2$ is the Euclidean plane $\mathbf{E}^2$ with an extra "line at infinity" adjoined, each point of which represents a parallel direction in $\mathbf{E}^2$. $\mathbf{P}^2$ has the virtue of simplicity: every pair of points determines a unique line which is topologically a circle (i.e., a simple closed curve), and every two lines meet at a unique point. Thus our question becomes: When is there a homeomorphism of $\mathbf{P}^2$ to itself that simultaneously straightens all the members of a finite family Γ of simple closed curves?

Certainly a necessary condition is that each of the curves be "nicely" embedded in the plane. More precisely, for each curve there must be some homeomorphism that takes $\mathbf{P}^2$ to itself and maps the curve to a straight line. In addition, every two of our curves must meet exactly once, and cross at their point of intersection, just as straight lines do. Are these two conditions sufficient? The answer is no, and a counterexample can easily be constructed using Desargues' theorem.

Desargues' theorem, one of the basic theorems of projective geometry, asserts that if the corresponding sides of two triangles meet at three collinear points, then the three lines joining corresponding vertices are concurrent: see Figure 1a. On the other hand, Figure 1b is an example of an arrangement for which Desargues' theorem fails: any homeomorphism of the plane to itself that mapped the ten curves in Figure 1b to straight lines would yield an arrangement of lines that violated Desargues' theorem. Hence there is no homeomorphism of the plane to itself that simultaneously straightens the ten curves of Figure 1b.

Let us look for a moment at our two necessary conditions. A straight line l in $\mathbf{P}^2$ does not separate $\mathbf{P}^2$, since any two points in $\mathbf{P}^2 \setminus \{l\}$ are connected by some path, perhaps one crossing the line at infinity. (In contrast to this, a "small" circle does separate $\mathbf{P}^2$.) Thus if there is to be a homeomorphism of $\mathbf{P}^2$ to itself which maps some simple closed curve l' to a straight line, then l' must also not separate $\mathbf{P}^2$, i.e., $\mathbf{P}^2 \setminus \{l'\}$ must be connected. It follows from Schoenflies' Theorem (see [**12**], for example) that the converse is true as well: If l' is a simple curve that does not separate $\mathbf{P}^2$, then there is a homeomorphism taking $\mathbf{P}^2$ to itself that maps l' to a

Some of the main results of this paper were presented at the Eighth Annual ACM Symposium on Computational Geometry in Berlin on June 11, 1992 [**5**].

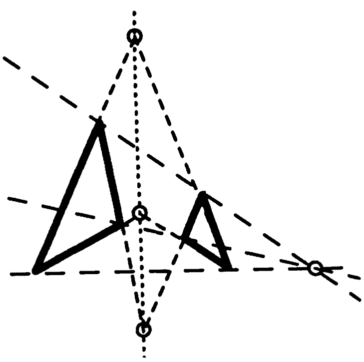

Figure 1a. Desargues' Theorem.

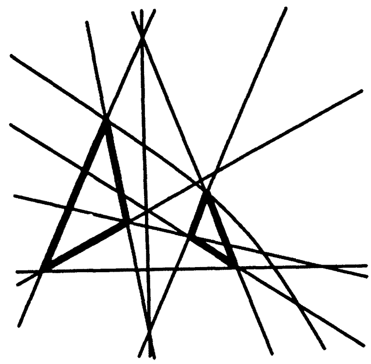

Figure 1b. A non-stretchable arrangement.

straight line. Such a simple closed curve is called a *pseudoline*. A finite family of pseudolines in $\mathbf{P}^2$, with the property that any two meet exactly once (and necessarily cross), is known as an *arrangement of pseudolines.*

To visualize an arrangement $\mathscr{A}$ of pseudolines in $\mathbf{P}^2$, one can model the projective plane as a circular disk with opposite points identified. For this purpose, remove some pseudoline $l^* \in \mathscr{A}$ from the projective plane. The remaining points then form a space (the Euclidean plane!) homeomorphic to an open circular disk in $\mathbf{E}^2$. Call the closure of the disk Δ. Each point on l^* corresponds to an antipodal pair of points on the circle $\partial\Delta$. A pseudoline in $\mathscr{A}$ other than l^* becomes a curve connecting antipodal points in Δ. Thus an arrangement of pseudolines in $\mathbf{P}^2$ corresponds to a family of Jordan arcs connecting antipodal points on a circle, every pair of arcs intersecting exactly once (or possibly at their endpoints); see Figure 2.

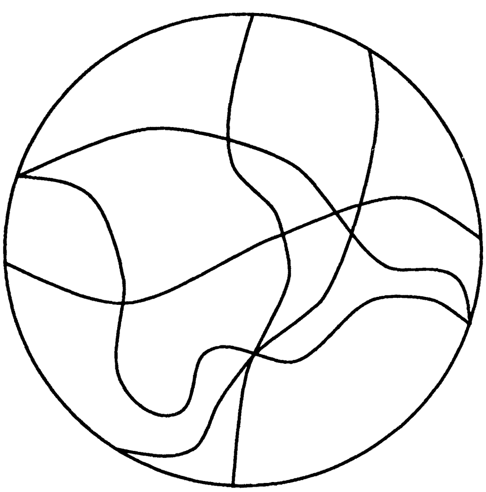

Figure 2. An arrangement of pseudolines in the disk model of $\mathbf{P}^2$.

Arrangements of pseudolines have been studied since the work of F. Levi **[8]**, who first pointed out that, in spite of their resemblance to arrangements of straight lines, they are topologically more general objects. B. Grünbaum **[6]** published an extensive monograph on arrangements in 1971, in which he answered many questions about line- and pseudoline-arrangements and posed many others. It turns out that the study of arrangements of pseudolines is equivalent to that of oriented matroids of rank 3; see [1] for a definition and good introduction to oriented matroids, and in particular for a discussion of their relationship to pseudoline arrangements.

As in any field of mathematics, we classify arrangements, i.e., we partition them into classes of similar or "isomorphic" ones. This is done by considering their topology: Two arrangements are called *isomorphic* if there is some homeomorphism of $\mathbf{P}^2$ to itself that maps the pseudolines in one arrangement to those in the other.

An arrangement $\mathscr{A}$ of pseudolines, like an arrangement of straight lines, induces a decomposition of the projective plane into a cell complex $\mathscr{C}(\mathscr{A})$, consisting of cells of dimension 2 ("faces"), 1 ("edges"), and 0 ("vertices"). An isomorphism between two arrangements induces a one-to-one correspondence between their cell complexes that preserves incidence, i.e., neighboring faces in one arrangement map to neighboring faces in the other. The converse is also true: Suppose $\mathscr{A}$ and $\mathscr{A}'$ are arrangements and there is a one-to-one correspondence between $\mathscr{C}(\mathscr{A})$ and $\mathscr{C}(\mathscr{A}')$ that preserves incidence. Patching together homeomorphisms between corresponding faces gives a homeomorphism of the plane to itself that maps the pseudolines of one arrangement to the pseudolines of the other. Thus isomorphism between arrangements is really just a combinatorial relationship that can be defined solely in terms of the cells and their incidences.

The question we posed at the beginning can now be restated as follows: Is every arrangement of pseudolines isomorphic to some arrangement of straight lines? The example above based on Desargues' theorem shows that the answer is "no". (G. Ringel showed [**15**] that even if we consider only arrangements with no multiple points, the answer still remains "no".) An arrangement of pseudolines that is isomorphic to some arrangement of straight lines is called *stretchable*. Unfortunately, determining whether an arrangement of pseudolines is stretchable turns out to be quite a hard problem (in fact, NP-hard: see [**10**, **11**, **17**].)

While not every arrangement of pseudolines is stretchable, arrangements of pseudolines nevertheless share many properties with arrangements of straight lines. Given any arrangement of n straight lines we can always add another line through any two given points not both on the same line to form an arrangement of $n + 1$ straight lines. The same property holds for arrangements of pseudolines [**8**].

This result, known as the Levi Enlargement Lemma, implies that any arrangement of n pseudolines can be extended to an arrangement of $n + 1$ pseudolines. However, an arrangement of n straight lines can be extended to a much larger and richer structure, the topological space consisting of *all* the lines in the plane. Each line is a "point" in this space, and a neighborhood of the line through two points consists of all the lines through nearby pairs of points.

Can every finite arrangement of pseudolines be embedded in some continuous family of pseudolines analogous to the set of all lines in the plane? Grünbaum posed this question in [**6**]. What should be the properties of such a family of pseudolines? First, this family should form a topological space analogous to the space of all lines in the plane. Second, this set of pseudolines, together with the set of points in the projective plane, should obey the basic incidence axioms of geometry: any two points should lie on a unique pseudoline, and any two pseudolines should intersect in a unique point. Third, the geometric and topological properties should be linked as they are for lines: as two pseudolines vary continuously, their point of intersection should vary continuously; as two points vary continuously, the unique pseudoline they define should vary continuously as well.

Such structures are known to exist, and have in fact been studied for nearly a century. They are known as *topological projective planes*. A topological projective plane Q, in the sense we will use the phrase, consists of $\mathbf{P}^2$ as its underlying point set, and a second topological space $L(Q)$ consisting of simple closed curves in $\mathbf{P}^2$ as its set of "lines"; these satisfy the following conditions:

1. for every two distinct points $p, q \in Q$ there is a unique curve $l(p, q) \in L(Q)$ containing p and q;
2. every two distinct curves $l, l' \in L(Q)$ intersect in exactly one point at which they cross;
3. $l(p, q)$ varies continuously as a function of p and q;
4. $l \cap l'$ varies continuously as a function of l and l'.

As before, we use the term *pseudolines* for the curves in $L(Q)$.

Just as the neighborhood of a straight line through two points consists of all the lines through nearby pairs of points, the neighborhood of a pseudoline consists of all pseudolines through nearby pairs of points. Since $\mathbf{P}^2$ is compact, this is equivalent to the topology induced by the Hausdorff metric on pseudolines: two pseudolines are within distance ϵ of each other if each point of each pseudoline is within ϵ of the other pseudoline; here the metric on $\mathbf{P}^2$ is the one coming from the standard metric on the sphere $\mathbf{S}^2$ when antipodal points are identified.

Topological planes were discussed by Hilbert in his seminal book *Foundations of Geometry* [**7**]. There, he gave the first example of a topological Euclidean plane in which Desargues' theorem failed to hold. F. R. Moulton subsequently gave a simpler example, now known as the Moulton plane [**13**], which was incorporated into later versions of Hilbert's book.

More recently, H. R. Salzmann [**16**] studied topological planes and their various axiomatizations. Among other results, he proved that, with hypotheses even weaker than the above conditions, $L(Q)$ will always be homeomorphic to the space of points of the projective plane $\mathbf{P}^2$. He also showed that the fourth condition, that $l \cap l'$ vary continuously as a function of l and l', is a consequence of the third condition, that $l(p, q)$ vary continuously as a function of p and q, and vice versa. Many other interconnections among properties of topological planes are given in Salzmann's paper.

Grünbaum's question can now be reformulated as follows: Given an arrangement $\mathscr{A}$ consisting of a finite number of pseudolines in the projective plane, is there some topological projective plane Q containing $\mathscr{A}$, i.e., a plane such that $\mathscr{A} \subset L(Q)$? If $\mathscr{A}$ were stretchable, this would be trivially true: take an isomorphic arrangement of straight lines, consider it as embedded in $\mathbf{P}^2$, and use the homeomorphism of $\mathbf{P}^2$ that straightens the pseudolines of $\mathscr{A}$ to define a new topological plane structure on $\mathbf{P}^2$ in which the "lines" are simply the inverse images of the straight lines of $\mathbf{P}^2$. But in general, if $\mathscr{A}$ is not stretchable, no such argument is available. Nevertheless, we will answer the question affirmatively in Section 2 below, by showing how to extend any arrangement of pseudolines to a topological projective plane; the solution will turn out, in fact, to be surprisingly simple.

Grünbaum asked yet another, more sweeping, question in "Arrangements and Spreads". Assuming that every finite arrangement of pseudolines can be embedded in some topological projective plane, is there a *single* topological plane that contains every finite arrangement of pseudolines up to isomorphism? Another way of posing this question is to extend our notion of stretchability. Recall that an arrangement of pseudolines is *stretchable* if it is isomorphic to some arrangement of lines in $\mathbf{P}^2$. For a topological projective plane Q, let us call an arrangement $\mathscr{A}$ of pseudolines in the projective plane *stretchable in* Q if it is isomorphic to some arrangement of pseudolines $\mathscr{A}' \subset L(Q)$. Then Grünbaum's question becomes: Is there some topological plane Q such that every arrangement of pseudolines is stretchable in Q?

We will show in Section 3, using the embedding theorem of Section 2, that the answer is again "yes". Grünbaum called such a topological plane, whose existence he conjectured, a *universal* topological plane.

2. FROM ARRANGEMENTS TO TOPOLOGICAL PLANES. One obvious approach to extending arrangements to topological planes is the repeated use of the Levi Enlargement Lemma, adding new pseudolines one at a time in an infinite process. This would only generate a countably infinite family of pseudolines, however, so one must then "complete" this set by taking some sort of limit. The problem with such an approach is that one may unwittingly introduce discontinuities in taking this limit. While it is conceivable that such a technique may work, no one, to our knowledge, has been able to give a construction along these lines.

Instead of adding pseudolines one at a time, our method will be to define pseudolines piecewise in different regions of the plane, and then to link the pieces together. As previously noted, an arrangement $\mathscr{A}$ of pseudolines splits the projective plane into faces. We will construct the topological plane by defining all the "pseudoline segments" traversing a given face and then showing how to join these segments to form pseudolines with the desired properties.

To carry out our construction, we need a simple fact about stretchability, proved first by J. Richter-Gebert **[14]** in the "uniform" case, i.e., where no three pseudolines are concurrent:

Lemma 1. *Let $\mathscr{A}$ be an arrangement of n pseudolines. If some face of $\mathscr{A}$ is bounded by at least $n - 1$ pseudolines, then $\mathscr{A}$ is stretchable, i.e., isomorphic to an arrangement of straight lines.*

The lemma is proved, without any assumption of uniformity, at the end of this section.

We now proceed with our construction. Let $\mathscr{A}$ be any arrangement of n pseudolines in $\mathbf{P}^2$. Fix some distinguished pseudoline $l^* \in \mathscr{A}$. This pseudoline will play a role in the topological projective plane similar to the role of the line at infinity in the standard model of $\mathbf{P}^2$.

For each face f of the arrangement $\mathscr{A}$, let L_f be the set of pseudolines bounding f. By Lemma 1, $L_f \cup \{l^*\}$ is stretchable. Let h_f be a homeomorphism of the projective plane to itself that maps the pseudolines in $L_f \cup \{l^*\}$ to straight lines (notice that it is possible that $L_f \cup \{l^*\} = L_f$). For each pair of distinct points p and q lying on different segments of ∂f, there is a straight line segment a in $h_f(f)$ connecting $h_f(p)$ to $h_f(q)$. $h_f^{-1}(a)$ is then an arc in f with endpoints p and q; see Figure 3. (As previously described, the arrangement $\mathscr{A}$ can be visualized as a set of Jordan arcs connecting antipodal pairs of points in a disk where l^* maps to the circle bounding the disk. Of course the "straight" lines shown in Figure 3b are not really *straight* in the disk model; they can, however, be taken to be arcs of circles joining antipodal pairs on the disk boundary—see **[6]**.)

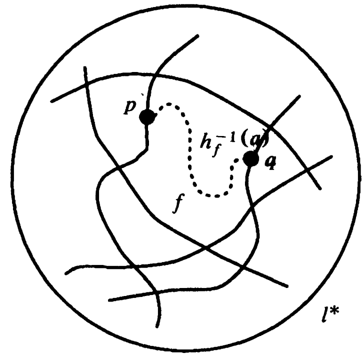

Figure 3a. Original face f.

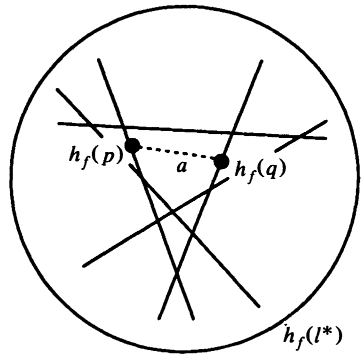

Figure 3b. A straightening of f.

Let Γ be the set of all such arcs over all faces of $\mathscr{A}$; the members of Γ form the "pieces" of the pseudolines we are going to construct.

Let γ be some arc of Γ lying in face f. $h_f(\gamma)$ is a line segment which is part of some straight line l meeting $h_f(l^*)$ at a point r "at infinity". Let $s \in l^*$ be $h_f^{-1}(r)$, and label γ with the point s. (One can think of s as the "slope" of arc γ even though γ may be far from linear, and even though this "slope" depends on the chosen straightening homeomorphism h_f.) Label every arc $\gamma \in \Gamma$ in this manner, and notice that if p is a point on l^*, then every arc with endpoint p has label p. If p is a point on the interior of some edge of f, where $p \in l$ and $l \in L_f$ and $p \notin l^*$, then every point of $l^* \setminus l$ occurs exactly once as a label of an arc in f with endpoint p. Finally, if p is a vertex of f, where $p \in l \cap l'$ and $l, l' \in L_f$ and $p \notin l^*$, then the labels of arcs in f with endpoint p form an arc on l^* between $l \cap l^*$ and $l' \cap l^*$.

The arcs in Γ can now be linked together to form pseudolines. For every point p lying on some pseudoline $l \in \mathscr{A} \setminus \{l^*\}$, let Γ_p be the set of arcs in Γ with endpoint p. Then for every arc $\gamma \in \Gamma_p$ with label s there is exactly one other arc $\gamma' \in \Gamma_p$, on the other side of l, with the same label. Join these two arcs to form a longer arc, and continue. Repeating this for every point p lying on some pseudoline other than l^*, we get a set $\tilde{\Gamma}$ of arcs. We claim that $L(Q) = \tilde{\Gamma} \cup \mathscr{A}$ is the set of pseudolines of a topological plane Q.

For the proof, it is useful to refer again to the disk model described above, in which each point $s \in l^*$ is replaced by an antipodal pair s^+, s^- on the circle $\partial\Delta$ bounding a disk Δ, and a pseudoline that intersects l^* at s becomes a curve with endpoints s^+ and s^-.

We first show that in this disk model the endpoints of every $l \in \tilde{\Gamma}$ constitute an antipodal pair in $\partial\Delta$. Start with any arc $\gamma_0 \in \Gamma$ that forms a segment of l. γ_0 has some label s. Arc γ_0 is separated from s^+ by some $k < n$ pseudolines of $\mathscr{A}$. One of the two arcs joined to γ_0 is therefore separated from s^+ by only $k - 1$ pseudolines; let this arc be γ_1. γ_1 also has label s. Repeating this argument k times gives an arc γ_k with label s which has endpoint s^+; thus l has s^+ as one endpoint. A similar argument shows that s^- is the other endpoint of l.

If two arcs have the same endpoints on $\partial\Delta$, i.e., the same labels, then it is clear that they can never intersect in the interior of Δ. On the other hand, it is also clear that two arcs in $L(Q)$ cannot intersect infinitely often, since they can meet at most once inside each face of $\mathscr{A}$. To prove that they intersect *exactly* once, we first establish a general lemma about arcs intersecting in the disk (cf. **[4]**).

Let l and l' be any two arcs in Δ connecting distinct antipodal endpoints s_1, s_3 and s_2, s_4, respectively; l and l' may intersect at more than one point. Let p be an isolated point of intersection of l and l' at which they cross. In other words, there is some small topological disk Δ^* containing p and no other point of intersection of l and l'. Arcs l and l' intersect $\partial\Delta^*$ at four points, s_1^*, s_2^*, s_3^*, and s_4^*, with s_i^* lying between p and s_i on l or l'. We say that p is a *proper* intersection point of l and l' if s_1, s_2, s_3, and s_4 occur in the same order around Δ (clockwise or counterclockwise) as s_1^*, s_2^*, s_3^*, and s_4^* do around Δ^*. (See Figure 4.)

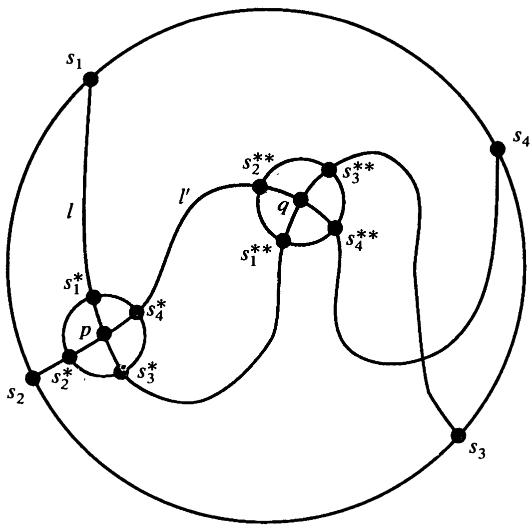

Figure 4. p is a proper intersection point; q is not.

The following lemma replaces the global condition that arcs intersect at precisely one point, at which they cross, by the local condition that every point of intersection is proper.

Lemma 2. *Two arcs connecting distinct antipodal points in the disk that intersect finitely often intersect at precisely one point, and cross there, if and only if every point of intersection of the two arcs is proper.*

Proof: Since the endpoints of each arc are antipodal, the first arc separates the endpoints of the second and thus the arcs must have at least one intersection point. If there is only one, that intersection must clearly be proper. On the other hand, if our two pseudolines l and l' intersect at more than one point, we can list the points of intersection in order along l. Let p and q be two successive points of intersection. Then it follows immediately from the definition that if p is proper, q is not. □

We now prove that every two arcs in $L(Q)$ intersect exactly once. Let l, $l' \in L(Q)$ be two arcs connecting distinct antipodal endpoints s_1, s_3 and s_2, s_4, respectively. Suppose p is an intersection point of l and l' lying in the interior of a face f. By construction, l and l' meet the boundary of f in four points, s_1^*, s_2^*, s_3^*, and s_4^*, where s_i^* lies between s_i and p on l or l'. The order of s_1, s_2, s_3, and s_4 around Δ agrees with the order of $h_f(s_1)$, $h_f(s_2)$, $h_f(s_3)$, and $h_f(s_4)$ around $h_f(\Delta)$. Similarly, the order of s_1^*, s_2^*, s_3^*, and s_4^* around f agrees with the order of $h_f(s_1^*)$, $h_f(s_2^*)$, $h_f(s_3^*)$, and $h_f(s_4^*)$ around $h_f(f)$. By construction, the intersection of $h_f(l)$ and $h_f(l')$ in $h_f(f)$ is proper; hence the intersection of l and l' in f must also be proper. A similar argument holds if p lies on the boundary of a face f. Thus, by Lemma 2, every two arcs in $L(Q)$ intersect exactly once.

For every point $p \in \mathbf{P}^2$ and every $s \in l^*$, there is a unique pseudoline in $L(Q)$ passing through p and s. This pseudoline varies continuously as a function of s, sweeping over the whole projective plane as s runs through l^*. Hence, for any $p, q \in \mathbf{P}^2$, there is some pseudoline in $L(Q)$ containing p and q. If two distinct pseudolines $l, l' \in L(Q)$ both contained p and q, then l and l' would intersect more than once. Thus there must be a unique pseudoline $l(p, q) \in L(Q)$ containing p and q.

Finally, the continuity conditions on $l(p, q)$ and $l \cap l'$ follow from the fact that continuity is a local property, and that locally our pseudolines are nothing but homeomorphic images of lines.

We have thus proved

Theorem 1. *Every arrangement of pseudolines in the projective plane can be extended to a topological projective plane.*

We conclude this section with the proof of Lemma 1 promised above.

Proof of Lemma 1: Let $\mathscr{A}$ be an arrangement of n pseudolines, at least $n - 1$ of which bound a face f of the arrangement, and let l be the nth pseudoline (or *any* one if all n bound f). l is stretchable, so there is some homeomorphism of the projective plane to itself that maps l to the line at infinity, l_∞. Thus, without loss of generality, we may assume that $l = l_\infty$. Let $\{l_1, l_2, \ldots, l_{n-1}\}$ be the remaining set of pseudolines, $\mathscr{A} \setminus \{l_\infty\}$.

If we remove l_∞ from the projective plane, we are left with a Euclidean plane, which we assume coordinatized. Each point $l_i \cap l_\infty$ can now be identified with some slope s_i in this plane. Without loss of generality, we can assume none of the slopes s_i is infinite. For each i, let l_i' be the line with slope s_i tangent to the unit circle, with l_i' chosen so that it passes above the unit circle if and only if l_i passes above f; see Figure 5. (l_i passes above f if there exists some suitably large y_0 such that l_i separates f from $(0, y)$ for all $y > y_0$.)

We claim that the pseudoline arrangement $\{l_1, l_2, \ldots, l_{n-1}\}$ is isomorphic to the straight line arrangement $\{l_1', l_2', \ldots, l_{n-1}'\}$ in the Euclidean plane. Clearly this will imply that the arrangements $\{l_1, l_2, \ldots, l_{n-1}, l_\infty\}$ and $\{l_1', l_2', \ldots, l_{n-1}', l_\infty\}$ are isomorphic in the projective plane.

The proof is by induction on the number of lines in the arrangements. It is trivially true for the arrangements $\{l_1\}$ and $\{l_1'\}$. Assume the arrangement $\mathscr{A}_k = \{l_1, l_2, \ldots, l_k\}$ is isomorphic to the arrangement $\mathscr{A}_k' = \{l_1', l_2', \ldots, l_k'\}$. Consider what happens when we add l_{k+1} and l_{k+1}' to $\mathscr{A}_k$ and $\mathscr{A}_k'$, respectively. Without loss of generality, assume l_{k+1} lies below face f. Orient l_{k+1} so that face f lies to its left. Pseudoline l_{k+1} first intersects the pseudolines of A_k lying above f whose slope is greater than s_{k+1} in order of increasing slope. l_{k+1} then intersects the pseudolines of A_k lying below f in order of increasing slope. Finally, l_{k+1}

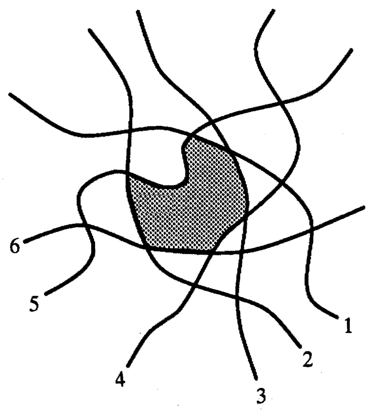

Figure 5a. Pseudolines around a face.

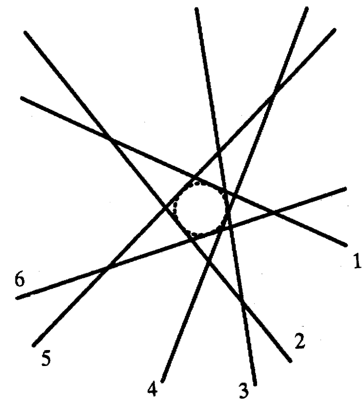

Figure 5b. Lines around the unit circle.

intersects the pseudolines of A_k lying above f whose slope is less than s_{k+1} in order of increasing slope. (If some pseudoline above f is "parallel" to l_{k+1}, we can consider it to be met by l_{k+1} after the rest.)

By construction, l'_{k+1} lies below the unit circle; orient it so that the unit circle lies to its left. Then l'_{k+1} also first intersects lines above the unit circle with slope greater than s_{k+1}, then lines below the unit circle, and finally the lines above the unit circle with slope less than s_{k+1}. The order in which l'_{k+1} meets them is also by increasing slope. Since l_i passes above f if and only if l'_i passes above the unit circle, and since the slopes of l_i and l'_i are equal, l_{k+1} and l'_{k+1} intersect corresponding pseudolines and lines in the same order. It follows that l_{k+1} and l'_{k+1} split corresponding faces in $\mathscr{A}_k$ and $\mathscr{A}'_k$ in exactly the same manner. Therefore the arrangements $\mathscr{A}_{k+1} = \{l_1, l_2, \ldots, l_k, l_{k+1}\}$ and $\mathscr{A}'_{k+1} = \{l'_1, l'_2, \ldots, l'_k, l'_{k+1}\}$ are isomorphic, and hence $\mathscr{A} \setminus \{l_\infty\}$ is isomorphic to the arrangement $\{l'_1, l'_2, \ldots, l'_{n-1}\}$ of straight lines in the Euclidean plane. □

3. UNIVERSAL TOPOLOGICAL PLANES. Recall that an arrangement $\mathscr{A}$ of pseudolines is *stretchable in* Q if it is isomorphic to some arrangement of pseudolines $\mathscr{A}' \subset L(Q)$. In the previous section we proved that for every arrangement $\mathscr{A}$ there is some topological projective plane Q in which $\mathscr{A}$ is stretchable. We will now show that there is a topological plane in which *every* arrangement is stretchable.

We first need to introduce a technique for "patching" parts of one topological plane into another. Let Q and Q' be two topological planes. Let l_1, l_2, and l_3 be pseudolines in Q that are not concurrent. l_1, l_2, and l_3 then decompose Q into four closed regions which we call *triangles*. Let τ be any one of these triangles. The *vertices* of triangle τ are the points $l_1 \cap l_2$, $l_2 \cap l_3$, and $l_1 \cap l_3$; see Figure 6.

Similarly, choose three pseudolines in Q' that are not concurrent, label them l'_1, l'_2, and l'_3, and let triangle τ' be one of the closed regions bounded by these three pseudolines. Let ϕ be a homeomorphism of τ' onto τ that maps the vertices of τ' to the corresponding vertices of τ; again, this will always exist by virtue of Schoenflies' theorem [**12**].

Define a new topological plane Q'', with a new set of pseudolines $L(Q'')$ chosen as follows. For each $l \in L(Q)$, if $l \cap \operatorname{int}(\tau) = \varnothing$, then let l belong to $L(Q'')$. Otherwise, $l \cap \tau$ is a connected arc with two endpoints, p and q. Let l' be the

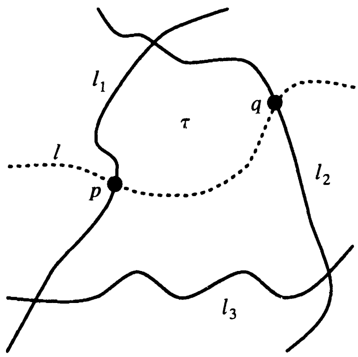

Figure 6a. Triangle τ.

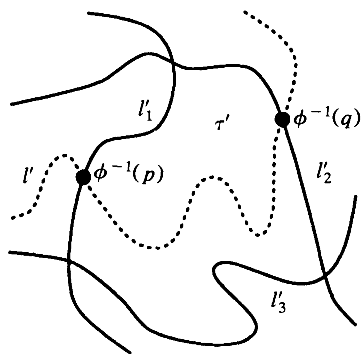

Figure 6b. Triangle τ'.

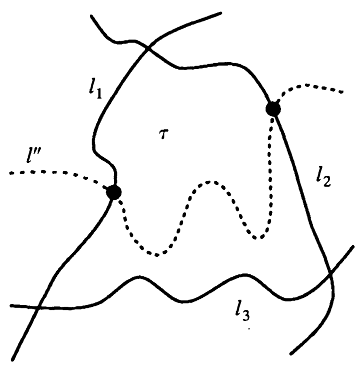

Figure 6c. Topological plane Q''.

unique pseudoline in Q' passing through $\phi^{-1}(p)$ and $\phi^{-1}(q)$. Replace $l \cap \tau$ by $\phi(l' \cap \tau')$ to form a new pseudoline l'', and let l'' belong to $L(Q'')$.

We must show that every two distinct pseudolines l''_0, $l''_1 \in L(Q'')$ intersect exactly once. But this is immediate from the fact that *outside* τ nothing has been altered, while *inside* τ two pseudolines meet if and only if their intersections with $\partial\tau$ interlace, a property which is preserved by the homeomorphism ϕ.

We can now prove

Theorem 2. *There exists a universal topological plane in which every arrangement of pseudolines is stretchable.*

Proof: Every arrangement of n pseudolines has at most $\binom{n}{2} + 1$ faces. Since the question of whether two arrangements are isomorphic depends only on the combinatorial structure of their associated cell complexes, there are only a finite number of isomorphism classes of arrangements of n pseudolines. Thus the set of isomorphism classes of arrangements of pseudolines of arbitrary size is countable.

Let $\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3, \ldots$ be a sequence of pseudoline arrangements such that every arrangement is isomorphic to some $\mathscr{A}_i$. Let Q_0 be any topological projective plane (for example $\mathbf{P}^2$), and let $\tau_1, \tau_2, \tau_3, \ldots$ be a sequence of pairwise disjoint triangles in Q_0. For each $i > 0$, construct the topological plane Q_i from the topological plane Q_{i-1} as follows. Embed $\mathscr{A}_i$ in some topological plane Q', using Theorem 1. Let τ' be some triangle in Q' containing all the intersection points of the pseudolines in $\mathscr{A}_i$. (The existence of such a triangle τ' follows by continuity, since we can always find a pseudoline avoiding all the intersection points in $\mathscr{A}_i$ and then take two other pseudolines sufficiently close to the first so that all the intersection points remain within a single region.) Replace triangle τ_i in Q_{i-1} by triangle τ' using the patching technique described above, to form the topological plane Q_i.

Let Q_∞ be the topological plane which is the limit of the topological planes τ_i. The pseudolines $L(Q_\infty)$ are formed from the original pseudolines in Q_0 by a (possibly infinite) sequence of local replacements. If two pseudolines $l, l' \in L(Q_\infty)$ intersected at two distinct points, p and q, then some corresponding pseudolines l_i and l'_i in some topological plane Q_i would also intersect at p and q, which is impossible. Thus Q_∞ is a topological plane. (The continuity conditions again follow as above, since they hold locally.)

Since all the intersections of pseudolines in $\mathscr{A}_i$ occurred inside τ', $\mathscr{A}_i$ is isomorphic to some arrangement $\mathscr{A}'_i \subset L(Q_i)$. Because the triangle τ_i is never modified after the ith stage of the construction, it follows that $\mathscr{A}_i$ is isomorphic to some arrangement in every Q_j, $j \geqslant i$, including Q_∞. Thus every arrangement is stretchable in Q_∞, i.e., Q_∞ is a universal topological plane. □

Even though any universal topological plane contains all pseudoline arrangements, up to isomorphism, not all universal topological planes are isomorphic. In fact, using the techniques above, different choices of triangles and patchings may be shown to lead to uncountably many non-isomorphic universal planes [**3**], answering another question posed in [**6**].

4. OTHER DIRECTIONS. If we start with an arrangement of straight lines in the projective plane and let h_f be the identity map for every face f, then our construction will generate the standard projective plane $\mathbf{P}^2$, with $L(\mathbf{P}^2)$ consisting of the usual straight lines. If h_f is not the identity, however, then even starting with a straight line arrangement we can generate a topological plane that is not isomorphic to the usual one.

In some sense, however, our construction does not generate topological planes whose pseudolines are too different from those in our original arrangement. An arrangement is called *k-piecewise linear* if each pseudoline in the arrangement is the union of at most k line segments. For any k, it turns out that there are pseudoline arrangements that are not isomorphic to any k-piecewise linear arrangement. (This can be shown, for example, by bounding the number of k-piecewise linear arrangements of n pseudolines using the Milnor-Thom theorem [**9, 18**] on the Betti numbers of solution sets of polynomial inequalities of a semi-algebraic set, and using the lower bound proved in [**2**] on the number of isomorphism classes of arrangements of n pseudolines.) On the other hand, if we start with an arrangement of n pseudolines and construct a topological plane Q by our methods, then any arrangement $\mathscr{A} \subset L(Q)$ can be shown to be isomorphic to some k-piecewise linear arrangement, where k depends only on n, the number of pseudolines in the original arrangement $\mathscr{A}$; this fact plays an essential role in the proof of [**3**].

In [**6**], Grünbaum discusses an object intermediate between an arrangement of pseudolines and a topological plane. A *spread* of pseudolines is a 1-parameter family of pseudolines, any two meeting once, with the property that every point

$s \in l_\infty$ has a unique pseudoline through it which varies continuously as a function of s. In [**6**], Grünbaum asked if every arrangement can be extended to a spread, a question that we answered affirmatively in [**4**]. The much stronger result proved in Theorem 1 above is easily seen to imply that proposition. But our methods do not give any insight into the possibility of extending a spread to a topological plane, and this seems to be an intriguing question.

Finally, arrangements of pseudolines can be generalized to arrangements of pseudoplanes in dimension 3 (or of pseudohyperplanes in arbitrary dimension for that matter). These pseudoplanes should be "nicely" embedded in $\mathbf{P}^3$, every two should intersect in a pseudoline of each, and every three should intersect in a single point. Can any arrangement of pseudoplanes be embedded in a continuous 3-parameter family of pseudoplanes, some sort of topological 3-space analogous to the topological space of planes in $\mathbf{P}^3$?

Just as we demanded of the points and pseudolines of a topological projective plane, the points, pseudolines and pseudoplanes of a topological projective 3-space should obey the standard incidence axioms of geometry, and their intersections should vary continuously. Desargues' theorem, however, now turns out to be a direct consequence of these conditions, instead of being an independent axiom as it was in the plane [7]. Since any such geometry in which Desargues' theorem holds is isomorphic to the standard one, any such topological projective 3-space turns out to be isomorphic to the usual $\mathbf{P}^3$. On the other hand, there are certainly arrangements of pseudoplanes that are not isomorphic to arrangements of ordinary planes. This shows that there is no straightforward generalization of Theorem 1 possible to arrangements of pseudoplanes.

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Misunderstanding

Ah, you are a mathematician,
they say with admiration
or scorn.

Then, they say,
I could use you
to balance my checkbook.

I think about checkbooks.
Once in a while
I balance mine,
just like sometimes
I dust high shelves.

From *Intersections: Poems by JoAnne Growney,*
Kadet Press, Bloomsburg, PA, 1993, p. 50.