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For most Convex Discs Thinnest Covering is not Lattice-like

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The *covering density* $\vartheta(C)$ of a convex disc C in the Euclidean plane is defined as the infimum of the densities of all coverings with congruent copies of C. The *lattice covering density* $\vartheta_L(C)$ of C is the infimum of the densities of all lattice coverings with C. The family of those convex discs for which $\vartheta(C) = \vartheta_L(C)$ plays a special role. In this paper we show that this class of convex sets is small in the following sense:

Theorem. *Let $\mathcal{C}$ be the space of all convex discs in the plane equipped with the Hausdorff metric. Let $\mathcal{N}$ be the subset of $\mathcal{C}$ consisting of all convex discs C for which $\vartheta(C) \neq \vartheta_L(C)$. Then $\mathcal{N}$ is an open dense set in $\mathcal{C}$.*

An analogous result for packings is proved in another paper of the first author [3]. The first result concerning packing properties of typical convex sets is due to Gruber [4] who showed that most convex bodies have surprisingly few neighbors in their densest lattice packing. For other properties of typical convex sets we refer to the survey papers of Gruber [5] and Zamfirescu [7].

The openness of $\mathcal{N}$ follows immediately from the continuity of the functionals $\vartheta(C)$ and $\vartheta_L(C)$. Let $\mathcal{Q}$ be the family of polygons P with

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the property that no two different pairs of vertices of P determine parallel lines. It is easily seen that $\mathcal{Q}$ is dense in the set of all convex polygons, hence it is dense in $\mathcal{C}$. Thus, to prove the theorem it suffices to show that $\vartheta(P) > \vartheta_L(P)$ for all $P \in \mathcal{Q}$.

Let us recall the notion of a p-hexagon introduced by W. Kuperberg [6]. A *p-hexagon* is a convex hexagon with a pair of opposite sides of equal length. Here "opposite" means separated by two sides, and degenerate cases in which one, two, or three of the sides consist of a single point are allowed. Each p-hexagon P admits a tiling with congruent copies of P, however there is no lattice-tiling with P unless P is centrally symmetric. If P is a p-hexagon contained in C and $\mathcal{T}$ is a tiling with P, then the corresponding copies of C form a covering with density $a(C)/a(P)$. Therefore, denoting by $h_p(C)$ the maximum area of a p-hexagon contained in C, we have

$$\vartheta(C) \leq a(C)/h_p(C). \tag{1}$$

We shall use the technique of generalized Dirichlet cells introduced by L. Fejes Tóth [3]. (See also Bambah and Rogers' paper [1] for a more accurate description of these cells). In case of lattice arrangements the construction can be considerably simplified so, for sake of completeness, we describe it in some detail.

Let Λ be a lattice such that the sets $\{C+g\}$, $g \in \Lambda$, constitute a covering. Suppose that the interiors of the discs C and $C+a$, $a \in \Lambda$ intersect. Then the boundary of the set $C \cap (C+a)$ can be divided into two arcs A_1 and A_2 with the common end points x and y such that A_1 belongs to $\operatorname{bd} C$ and A_2 belongs to $\operatorname{bd}(C+a)$. The arcs $A_1 - a$ and $A_2 - a$ intersect in the points $x-a$ and $y-a$, they belong to $\operatorname{bd}(C-a)$ and $\operatorname{bd} C$, respectively and together they form the boundary of the set $(C-a) \cap C$. Let $\overline{C}$ be the set obtained from C by replacing the arcs A_1 and $A_2 - a$ by the straight segments xy and $(x-a)(y-a)$. Then the sets $\{\overline{C}+g\}$, $g \in \Lambda$, form a lattice covering such that $\operatorname{int} \overline{C} \cap \operatorname{int}(\overline{C}+a) = \emptyset$. Repeating this construction we obtain in finitely many steps a convex subset $\widetilde{C}$ of C such that $\{\widetilde{C}+g\}$, $g \in \Lambda$, is a lattice tiling. It follows that $\widetilde{C}$ is a, possibly degenerate, centrally symmetric hexagon. Hence we get

$$\vartheta_L(C) = a(C)/h^*(C), \tag{2}$$

where $h^*(C)$ denotes the maximum are of a centrally symmetric hexagon contained in C. In view of (1) and (2) we have to prove only the following

Lemma. *For any $P \in \mathcal{Q}$ we have*

$$h_p(P) > h^*(C).$$

Proof. Consider a polygon $P \in \mathcal{Q}$ and let H be a centrally symmetric hexagon contained in P. We shall show that there is a p-hexagon $\widetilde{H}$ such that $a(H) < a(\widetilde{H})$. We suppose, indirectly, that this is not true, and obtain a contradiction.

If H is degenerate, that is H is a parallelogram, then $H \neq P$, so choosing a point $p \in P \backslash H$, $\widetilde{H} = \operatorname{conv}(\{p\} \cap H)$ is a (degenerate) p-hexagon with the required property. Thus we have to consider only the case when $H = h_1 h_2 \ldots h_6$ is nondegenerate. If a vertex of H is contained in $\operatorname{int} P$, then there is a point $p \in \operatorname{int}(P \backslash H)$ such that $H \subset \operatorname{conv}(\{p\} \cap H)$. Again, $\widetilde{H} = \operatorname{conv}(\{p\} \cap H)$ is a p-hexagon such that $a(H) < a(\widetilde{H})$. Therefore we assume that

(i) *all vertices of H lie on* $\operatorname{bd} P$.

Next we assume that

(ii) *no three consecutive vertices h_{i-1}, h_i and h_{i+1} of H are in a position such that h_i is an interior point of a side s of P not parallel to the line $h_{i-1}h_{i+1}$.*

Otherwise we replace h_i by an appropriate point $\tilde{h}_i$ of s such that

$$a(h_{i-1}\tilde{h}_i h_{i+1}) > a(h_{i-1}h_i h_{i+1}).$$

We obtain thereby a p-hexagon $\widetilde{H} = h_1 \ldots \tilde{h}_i \ldots h_6$ contained in P with $a(\widetilde{H}) > a(H)$.

In particular, we assume that

(iii) *if a vertex of H is interior to a side of P, then the vertex of H opposite to it is a vertex of P.*

For, if both h_1 and h_4, say, are interior points to sides of P, then the diagonals h_6h_2 and h_3h_5 of H are parallel but the sides of P containing h_1 and h_4 are not. Thus we have a situation excluded above.

The proof of the Lemma will be complete by showing that there is no centrally symmetric hexagon satisfying properties (i) to (iii). Since $P \in \mathcal{Q}$, no six vertices of P can form a centrally symmetric hexagon. Therefore there is a vertex of H which is an interior point of a side of P. According to the above, the vertex of H opposite to this vertex is a vertex of P. It follows immediately that there are two consecutive vertices of H, h_1 and h_2 say, such that h_1 is a vertex of P and h_2 is an interior point to some side of P. We conclude by (ii) that h_3 is interior to a side of P, and by (iii) that h_6 and h_5 are vertices of P. By the central symmetry of H the diagonal h_2h_4 of H is parallel to h_1h_5. On the other hand, h_1h_5 is a diagonal of P, and by $P \in \mathcal{Q}$, the side of P containing h_3 is not parallel to this diagonal. Thus the vertices h_2, h_3 and h_4 are in a position contradicting assumption (ii).

This completes the proof of the Lemma and simultaneously the proof of our Theorem.

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