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On Some Questions about Convex Surfaces

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Introduction

Chapter A35 in the very enjoyable book [4] of CROFT, FALCONER and GUY treats geodesics on more or less smooth convex surfaces in $\mathbb{R}^3$. Our aim here is to answer a few questions mentioned there.

So for example the 43 years old problem of GÖTZ and RYBARSKI ([6], p. 301–302): Is the sphere the only surface for which whenever points can be joined by two distinct segments (i.e., shortest paths between two points), then they can be joined by an infinity of segments? We shall show here that every such surface is a "Wiedersehensfläche", i.e., a surface on which all geodesics from an arbitrary point meet again, the lengths of the geodesic arcs up to that meeting point being all equal (the term was invented by BLASCHKE). GREEN proved BLASCHKE's conjecture that all C^3 Wiedersehensflächen are spheres.

STEINHAUS [10] showed that there always exist at least two distinct segments from a point to any farthest point on a C^3 surface homeomorphic to S^2. This is also an easy consequence of the more recent well-known Aleksandrov-Toponogov theorem, which works for any C^3 variety (and in any dimension). We shall prove it for any C^1 convex surface. For arbitrary convex surfaces this is no longer true, as the example of a long thin pyramid shows.

STEINHAUS also asked what can be said qualitatively about the set of all farthest points from a given point on a convex surface, observing that it may not be connected. It has been pointed out by ALEKSANDROV that an (intrinsic) circle may be homeomorphic to any compact subset of the Euclidean circle (among other possibilities). The set of farthest points, a special circle on the surface, may be homeomorphic to any compact subset of the line, as well shall see. Each of its components must be a point or a Jordan arc. Its Hausdorff dimension is as expected at most 1 and its 1-dimensional Hausdorff measure at most πr_x, where r_x is the distance from the given point x.

The last section is devoted to the well-known conjecture of HILBERT and COHN-VOSSEN [8] claiming that every surface on which all geodesics from an arbitrary point meet again in another point is a sphere.

We shall make extensive use of the methods of ALEKSANDROV [1]. In most cases we will precisely refer to the used results from [1]. But generally speaking knowledge of large parts of [1] would be of much help for the reader.

Thanks are due to the referee for his or her comments.

On multijoined points

Let $S \subset \mathbb{R}^3$ be an arbitrary (closed) convex surface, i.e., the boundary of an open bounded convex set, and denote by ϱ its intrinsic metric. Let $x \in S$. A point $y \in S$ is called here multijoined to x if there are at least two segments from x to y.

It is easily seen that the set of all points y admitting at least 3 segments from x to y is at most countable. This has been independently observed by P. GRUBER.

The set C_x of all points of S multijoined conjugate to x is small from both the measure theoretic and Baire category points of view. Indeed, we proved in [14] that C_x is σ-porous and therefore of (2-dimensional) measure 0 and of first category. We are going to show here that C_x is always connected. This was known to differential geometers under stronger smoothness assumptions and conjectured for arbitrary convex surfaces in [15].

Theorem 1. *For any point x on a convex surface $S \subset \mathbb{R}^3$ the set C_x is arcwise connected.*

The proof of the theorem makes use of the following simple lemma.

Lemma 1. *Assume that a, b, y, z belong to a convex surface $S \subset \mathbb{R}^3$ and are all distinct (except possibly for a, b). If the distinct segments Σ_{ay} from a to y and Σ_{by} from b to y are equally long and the distinct segments Σ_{az} from a to z and Σ_{bz} from b to z are equally long too, then*

$$\Sigma_{ay} \cap \Sigma_{bz} = \Sigma_{az} \cap \Sigma_{by} = \{a\} \cap \{b\}.$$

Proof. It suffices to prove the second equality. Suppose, on the contrary, $c \in \Sigma_{az} \cap \Sigma_{by} \setminus \{a, b\}$. Since

$$\varrho(a, c) + \varrho(c, y) \geq \varrho(a, y) = \varrho(b, y),$$

it follows that $\varrho(a, c) \geq \varrho(b, c)$. Analogously, $\varrho(b, c) \geq \varrho(a, c)$, whence $\varrho(a, c) = \varrho(b, c)$.

Let Σ_1 be the subsegment of Σ_{az} from a to c and Σ_2 the subsegment of Σ_{by} from c to y. Since

$$\varrho(a, c) + \varrho(c, y) = \varrho(b, c) + \varrho(c, y) = \varrho(b, y) = \varrho(a, y),$$

$\Sigma_1 \cup \Sigma_2$ is a segment, different from Σ_{by} but having with it the common arc Σ_2, in contradiction with basic properties of segments on convex surfaces (see [1], p. 84–85). Since $a \in \Sigma_{by}$ only if $a = b$, because $\varrho(a, y) = \varrho(b, y)$, and, analogously, $b \in \Sigma_{az}$ only if $a = b$, the second equality is verified.

Proof of Theorem 1. If C_x is a single point there is nothing to prove. If $y, z \in C_x$ we choose two segments Σ_y^1, Σ_y^2 from x to y and another two Σ_z^1, Σ_z^2 from x to z. The mentioned basic properties of segments on convex surfaces ([1], p. 84–85) imply that

$$\Sigma_y^1 \cap \Sigma_y^2 = \{x, y\}, \qquad \Sigma_z^1 \cap \Sigma_z^2 = \{x, z\},$$

and Lemma 1 (with $x = a = b$) yields

$$(\Sigma_y^1 \cup \Sigma_y^2) \cap (\Sigma_z^1 \cup \Sigma_z^2) = \{x\}.$$

Thus the four segments divide the surface into three domains (i.e., open connected sets), one of which has all four on its boundary. Let D be this domain. We may assume that a

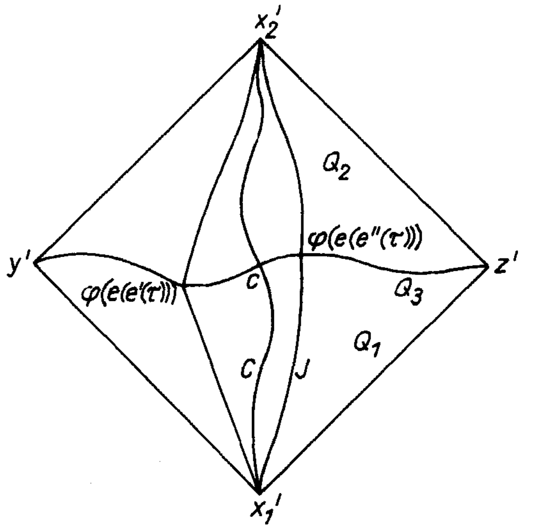

Fig. 1

small circle Γ (in the intrinsic metric of S) of centre x and radius ε meets the segments in the order Σ_y^1, Σ_y^2, Σ_z^2, Σ_z^1. Let Γ_y, Γ_2, Γ_z, Γ_1 be the arcs in which the preceding segments divide Γ (Γ_y between Σ_y^1 and Σ_y^2, Γ_2 between Σ_y^2 and Σ_z^2, etc.). Let Q be a (full, topologically closed) square of vertices x_1', y', x_2', z' in the Euclidean plane (see Figure 1). If Q' denotes Q with identified vertices x_1', x_2', then there obviously exists a homeomorphism φ between the closure $\bar{D}$ of D and Q' such that $x_1' = x_2' = \varphi(x)$, $y' = \varphi(y)$ and $z' = \varphi(z)$. This obviously induces a multifunction, which will also be denoted by φ, from $\bar{D}$ to Q (with $\varphi(x) = \{x_1', x_2'\}$). Equip $Q \setminus \{x_1', x_2'\}$ with the distance given by the length of the shortest path between the corresponding points in $\bar{D}$, which does not contain x. Define the distance between a point in $Q \setminus \{x_1', x_2'\}$ and x_i' to be the length of the shortest path from the corresponding point of $\bar{D}$ to x crossing Γ_i for ε arbitrarily small. Also, let the distance between x_1' and x_2' be the length of the shortest closed curve through x, not contractible in $\bar{D}$. Denote by δ this metric of Q and put

$$Q_1 = \{u \in Q : \delta(u, x_1') < \delta(u, x_2')\},$$
$$Q_2 = \{u \in Q : \delta(u, x_1') > \delta(u, x_2')\},$$
$$Q_3 = \{u \in Q : \delta(u, x_1') = \delta(u, x_2')\}.$$

Let τ_y, τ_z be the tangent directions of Σ_y^1, Σ_z^1 in x. They belong to the closed Jordan curve $T_x \subset S^2$ of all tangent directions at x. Let I be the arc on T_x from τ_y to τ_z not containing the tangent directions of Σ_y^2, Σ_z^2 in x. Now, a direction $\tau \in T_x$ is called singular if no segment starts at x in direction τ ([1], p. 213). For any nonsingular $\tau \in I$, let $e(\tau)$ be the endpoint of the maximal (by inclusion) segment $\Sigma(\tau)$ in S starting at x in direction τ. If τ is singular or $\varphi(e(\tau)) \in Q_1$, let $e'(\tau)$, $e''(\tau)$ be the endpoints of the maximal (by inclusion) open arc in T_x containing τ such that $\varphi(e(\sigma)) \in Q_1$ for any nonsingular σ in the arc.

Suppose $e(e'(\tau)) \neq e(e''(\tau))$ for some $\tau \in I$. Of course $\varphi(e(e'(\tau)))$ and $\varphi(e(e''(\tau)))$ are distinct and belong to Q_3. Consider the following four segments: $\varphi(\Sigma(e'(\tau)))$, a segment from $\varphi(e(e'(\tau)))$ to x_2', another segment from x_2' to $\varphi(e(e''(\tau)))$, and $\varphi(\Sigma(e''(\tau)))$. Their union is, by Lemma 1, a closed Jordan curve J. Let C be a Jordan arc from x_1' to x_2' whose interior points lie in the Jordan domain bounded by J and included in Q. Since $x'_1 \in Q_1$ and $x'_2 \in Q_2$, there must be a point c in $C \cap Q_3$. Then, for any segment $\Sigma \subset Q$ from x'_1 to c, the segment $\varphi^{-1}(\Sigma) \subset S$ must have a tangent direction σ between $e'(\tau)$ and $e''(\tau)$, while $\varphi(e(\sigma)) \notin Q_1$, a contradiction. Hence $e(e'(\tau)) = e(e''(\tau))$. Then the map $f: I \to S$ defined by

$$f(\tau) = \begin{cases} e(\tau) & \text{if} \quad \varphi(e(\tau)) \in Q_3\,, \\ e(e'(\tau)) & \text{otherwise}\,, \end{cases}$$

is continuous (use [1], p. 76) and f provides an arc whose points are all multijoined to x.

On farthest points

We shall prove here that the set F_x of all farthest points on S from $x \in S$ is strongly related to the set C_x: In any case $F_x \subset \overline{C_x}$, often $F_x \subset C_x$.

Theorem 2. *Let $S \subset \mathbb{R}^3$ be a convex surface and $x \in S$. Then $F_x \subset \overline{C_x}$. Any angle between two tangent directions at a point $y \in F_x$ measuring (on the tangent cone) more than π contains the tangent direction of a segment from y to x. So if the full angle of S at y is larger than π, then $y \in C_x$, and if S is differentiable at y and there are only two segments from x to y, then these have opposite tangent directions at y.*

Again we establish a lemma before proving the theorem.

Lemma 2. *Let Γ be a Jordan closed curve on a convex surface $S \subset \mathbb{R}^3$ and let $a, b, x \in S \setminus \Gamma$. Suppose that between every point of Γ and x there is a unique segment, and a, b do not belong to any such segment. Then Γ does not separate a from b.*

Proof. Suppose, on the contrary, that Γ separates a from b. Denote by Θ the union oF all segments joining points in Γ with x.

On one hand, since $a, b \notin \Theta$, but $\Gamma \subset \Theta$, the set Θ is not contractible.

On the other hand, if $p(v, r)$ denotes the point of the segment (supposed unique) from x to v, at distance r from v, the function p is continuous in both variables (use, for example, (10.5), (10.5'), (11.3) in [3]). Now, there is indeed a unique segment from x to v for any $v \in \Theta$. Then the homotopy $H: \Theta \times [0, 1] \to \Theta$ defined by

$$H(v, t) = p(v, t\varrho(x, v))$$

shows (see [5], p. 362) that Θ is contractible. A contradiction is found.

Proof of Theorem 2. Let Γ_ε be a small circle of radius ε around y, homeomorphic to S^1 (whose existence is guaranteed for ε small enough, see [1], p. 383). Suppose that $\Gamma_\varepsilon \cap C_x = \emptyset$. Then, by Lemma 2, every point on S separated from y by Γ_ε lies on the segment from x to some point of Γ_ε. Let Σ_u denote the segment from x to an arbitrary point $u \in \Gamma_\varepsilon$. Also, let Σ be a segment from x to y and consider $s \notin \Sigma$. If $\Gamma_\varepsilon \cap C_x = \emptyset$ for arbitrarily small $\varepsilon > 0$, let ε be so that Γ_ε separates s from y and consider the point $u \in \Gamma_\varepsilon$ with $s \in \Sigma_u$. By taking a sequence of numbers ε converging to 0, we get a sequence of segments with x as an endpoint, all containing s. This sequence converges to a segment from x to y containing s. Thus $y \in C_x$. Otherwise, if $\Gamma_\varepsilon \cap C_x \neq \emptyset$ for a sequence of numbers ε converging to 0, obviously $y \in \overline{C_x}$.

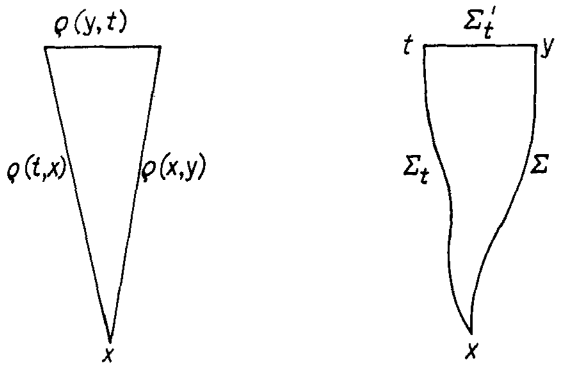

Fig. 2

Suppose that there is an angle T at y between τ_1 and τ_2 measuring on the tangent cone at y more than π, but not containing the tangent direction of any segment from y to x. Let τ_0 be the middle point of T (viewed as an arc of the rectifiable curve T_x). Clearly, the angles from τ_1 and τ_2 to τ_0 measure more than $\pi/2$. Let Σ' be a segment with an endpoint in y and with a tangent direction τ' at y so close to τ_0 that the angles from τ_1 and τ_2 to τ' are still larger than $\pi/2$. The existence of Σ' is guaranteed by the fact that the singular directions form a set of measure 0 (see [1], p. 213). Let $t \in \Sigma' \setminus \{y\}$ (see Figure 2). Of course, $\varrho(x, t) \leq \varrho(x, y)$. For $t \to y$ choose the segment Σ_t from t to x so that Σ_t converges to an arc Σ. Then Σ is a segment from y to x and therefore its tangent direction σ at y is not in T. Hence the angle from σ to τ' is larger than $\pi/2$. Consider the Euclidean triangle with side lengths $\varrho(x, y)$, $\varrho(y, t)$, $\varrho(t, x)$. Its angle α opposite to the side of length $\varrho(t, x)$ is not larger than its angle opposite to the side of length $\varrho(x, y)$ and therefore smaller than $\pi/2$. The angle between σ and τ' is smaller than $\alpha + \omega$; here ω is the curvature of the triangle with sides Σ, Σ'_t, Σ_t, where Σ'_t is the subarc of Σ' from y to t (see [1], p. 215). Since $\omega \to 0$ as $t \to y$,

$$\limsup_{t \to y} (\alpha + \omega) \leq \pi/2,$$

whence the angle between σ and τ' is at most $\pi/2$ and a contradiction is obtained.

The remaining assertions of the statement follow immediately.

The problem of GÖTZ and RYBARSKI

Now we pass to the mentioned problem of GÖTZ and RYBARSKI [6]. The results in the previous sections will be useful here.

Theorem 3. *Let $S \subset \mathbb{R}^3$ be a convex surface such that whenever points can be joined by two distinct segments then they can be joined by three distinct segments. Then S is a Wiedersehensfläche.*

Proof. Let $x \in S$ and suppose that C_x contains two points y, z. Then, by Theorem 1, C_x includes a whole arc A from y to z. Since there are only countably many points in C_x joined with x by at least three segments, for many points in A this cannot happen, in contradiction with the hypothesis. So C_x contains a single point y. By Theorem 2, y must be the unique farthest point of S from x. Indeed, if $y' \in F_x \setminus \{y\}$, then $y' \notin C_x$; in this case Theorem 2 tells us that $y' \in \overline{C_x} \setminus C_x$ which yields the infinity of C_x and a contradiction is obtained. For any nonsingular tangent direction τ at x let again $e(\tau)$ denote the other endpoint of the maximal segment starting at x in direction τ.

Suppose that $e(\tau) \neq y$ for some nonsingular τ. Let Γ be an intrinsic circle on S of centre $e(\tau)$ and with a radius small enough to guarantee that Γ is a Jordan curve (see [1], p. 383) separating $e(\tau)$ from both x and y. Since $e(\tau)$, y do not belong to any segment from a point of Γ to x, Lemma 2 implies that $e(\tau)$, y are not separated by Γ, and a contradiction is found again.

Hence $e(\tau) = y$ for all nonsingular $\tau \in T_x$. Because every tangent direction $\sigma \in T_x$ is the limit of a sequence of nonsingular directions, some subsequence of the corresponding sequence of segments from x to y converges to a segment from x to y having σ as tangent direction at x (use [1], p. 158). Hence σ is not singular.

Therefore for every $\tau \in T_x$ we have $e(\tau) = y$, which proves that S is a Wiedersehensfläche.

Corollary. *Each C^3 convex surface in $\mathbb{R}^3$ such that whenever points can be joined by two distinct segments then they can be joined by three distinct segments is a sphere.*

This follows from Theorem 3 together with GREEN's result in [7].

A question of STEINHAUS

We now turn to STEINHAUS' question about the set of all farthest points from some given point of a convex surface $S \subset \mathbb{R}^3$. The set F_x of all farthest points from $x \in S$ is a circle, the largest possible with centre at x. Let $C(x, r)$ denote the circle

$$\{y \in S : \varrho(x, y) = r\}$$

of centre x and radius r. Also let $r_x = \varrho(x, z)$, where $z \in F_x$. Clearly $C(x, r) \neq \emptyset$ if and only if $0 \leq r \leq r_x$, and $C(x, r_x) = F_x$. We denote by μ_α the α-dimensional Hausdorff measure. It is well-known that $\mu_1 C(x, r) \leq 2\pi r$. So $\mu_2 C(x, r) = 0$. The following easy proposition confirms this and applies of course in particular to F_x. For a definition and applications of porosity and strong porosity, see [12], [13].

Proposition. *For any point $x \in S$ and any $r \geq 0$, $C(x, r)$ is strongly porous.*

Proof. Let $y \in C(x, r)$ and consider a segment Σ from x to y. For any point $z \in \Sigma \setminus \{y\}$ the open ball B of centre z and radius $\varrho(z, y)$ is disjoint from $C(x, r)$. Indeed, for every point $w \in B$,

$$\varrho(x, w) \leq \varrho(x, z) + \varrho(z, w) < \varrho(x, z) + \varrho(z, y) = r$$

and therefore $w \notin C(x, r)$. Thus $C(x, r)$ is strongly porous.

By Lebesgue's density theorem, every porous set has measure 0.

More information on the dimension and measure of F_x is provided by the next theorem. For a definition and further facts on Hausdorff measures, see [9].

In the proof of the next theorem we shall use the following notation. If the segments $\Sigma^*, \Sigma^{**} \subset S$ have an endpoint b in common, let $(\Sigma^* b \Sigma^{**})$ denote the measure (on the tangent cone) of the (smaller) angle between Σ^* and Σ^{**} at b. Also, for $x \in S$, let $\nu(x) \subset S^2$ denote as usual the spherical image of x.

Theorem 4. *For any point $x \in S$, the Hausdorff dimension of F_x is at most* 1 *and*

$$\mu_1 F_x \leq \pi r_x .$$

Proof. The set of all points in F_x which are isolated or conical is at most countable. Let F^* be its complement in F_x. We only have to prove that $\mu_1 F^* \leq \pi r_x$.

Let $a, a_n \in F_x$ be such that $a_n \neq a$ and $a_n \to a$. Of course $\mu_2 \nu(a_n) \to 0$ as $n \to \infty$, whence the measure of the full angle of the tangent cone at a_n tends to 2π. So, by Theorem 2, from some index on, for every n there are two segments Σ_n', Σ_n'' from a_n to x; we choose them so that any other segment from a_n to x is separated from a by $\Sigma_n' \cup \Sigma_n''$ (see Figure 3). Then we may assume (take a subsequence if necessary) that $\Sigma_n' \to \Sigma'$ and $\Sigma_n'' \to \Sigma''$, say. Let Σ_n be some segment from a to a_n. Since $\varrho(x, a_n)$ is constant for $n \in \mathbb{N}$, $(\Sigma' a \Sigma_n)$, $(\Sigma'' a \Sigma_n)$, $(\Sigma_n' a_n \Sigma_n)$, $(\Sigma_n'' a_n \Sigma_n)$ converge all to $\pi/2$ (see [1], p. 381–382). So Σ' and Σ'' make an angle (the angle toward infinitely many a_n's, not the smaller one, but possibly both of them) equal to π at a. (This implies that the full angle of S at a measures at least π and that if it is precisely π then $\Sigma' = \Sigma''$.)

Now take $a \in F^*$. We have $(\Sigma' a \Sigma'') = \pi$. Even though there might be more segments from a to x, only Σ' and Σ'' are opposite at a. We associate these two segments to any point $a \in F^*$. Fix a certain sense on the rectifiable curve $T_x \subset S^2$ of all tangent directions at x. For any point $a \in F^*$, let $\alpha(a)$ and $\alpha'(a)$ be the tangent directions at x of the segments associated to a. That $\alpha(a)$ and $\alpha'(a)$ divide T_x into two arcs of equal lengths may happen for one point $a = a_0$ at most, because segments associated to different points of F^* cannot cross each other. For any point $a \in F^* \setminus \{a_0\}$ choose $\alpha(a)$ and $\alpha'(a)$ on T_x such that the arc of T_x from $\alpha(a)$ to $\alpha'(a)$ in the chosen sense is the smaller one. Let Δ denote the distance along T_x and set

$$A = \{\alpha(a) : a \in F^*\}, \qquad A' = \{\alpha'(a) : a \in F^*\},$$
$$A_m = \{\alpha(a) : \Delta(\alpha(a), \alpha'(a)) \geq m^{-1}\},$$
$$A_m' = \{\alpha'(a) : \Delta(\alpha(a), \alpha'(a)) \geq m^{-1}\}.$$

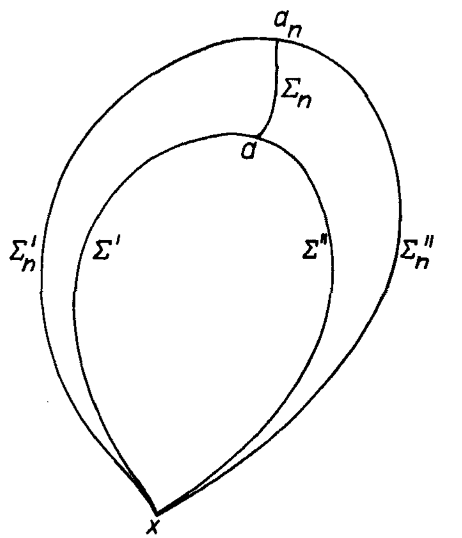

Fig. 3

Suppose now $\alpha(a_n) \to \beta$ on T_x and $\alpha(a_n) \in A_m$ for all n. Then $\alpha'(a_n) \to \beta'$ for some point β' such that the arc from β to β' in the chosen sense is not the larger one and $\Delta(\beta, \beta') \geq m^{-1}$. Suppose that the arcs of T_x from β to β' are not equally long. Then a suitable subsequence of $\{a_n\}_{n=1}^{\infty}$ converges to some point a and $\alpha(a) = \beta$, $\alpha'(a) = \beta'$ or a is a conical point. If a is a conical point, then $\beta \notin A \cup A'$. If not, $\beta \in A_m$ and this excludes $\beta \in A'_m$. Suppose now that the arcs of T_x between β and β' are equally long. Then a suitable subsequence of $\{a_n\}_{n=1}^{\infty}$ converges to a_0 and either $\alpha(a_0) = \beta$, $\alpha'(a_0) = \beta'$ or $\alpha(a_0) = \beta'$, $\alpha'(a_0) = \beta$.

Thus $\overline{A_m} \cap A' \subset \{a'(\alpha_0)\}$, whence

$$\left(\bigcup_{m=1}^{\infty} \overline{A_m}\right) \cap A' \subset \{\alpha'(a_0)\}\,.$$

Since $T = \bigcup_{m=1}^{\infty} \overline{A_m}$ and $T' = T_x \setminus T$ are complementary Borel sets in T_x,

$$\lambda T + \lambda T' = \lambda T_x\,,$$

and, because $A \subset T$ and $A' \subset T' \cup \{\alpha'(a_0)\}$,

$$\lambda A + \lambda A' \leq \lambda T_x\,,$$

where λ denotes the Lebesgue outer measure on T_x. Since $\lambda T_x \leq 2\pi$, for one of the sets A, A', say for A, we have $\lambda A \leq \pi$.

For any $\varepsilon > 0$ and $\delta > 0$ there is a covering $\{\widetilde{A}_i\}_{i=1}^{\infty}$ of A with diam $\widetilde{A}_i < \delta/r_x$ and

$$\sum_{i=1}^{\infty} \operatorname{diam} \widetilde{A}_i < \lambda A + \varepsilon/r_x\,.$$

Then $\{\alpha^{-1}(\widetilde{A}_i)\}_{i=1}^{\infty}$ is a covering of F^* and, for any pair of points a, b in $\alpha^{-1}(\widetilde{A}_i)$,

$$\varrho(a, b) \leq r_x\,\Delta(\alpha(a), \alpha(b)) \leq r_x \operatorname{diam} \widetilde{A}_i < \delta$$

(for the first inequality use Aleksandrov's convexity condition, [1], p. 47) and

$$\sum_{i=1}^{\infty} \operatorname{diam} \alpha^{-1}(\widetilde{A}_i) \leq r_x \sum_{i=1}^{\infty} \operatorname{diam} \widetilde{A}_i < r_x \lambda A + \varepsilon\,.$$

This yields $\mu_1 F^* \leq \pi r_x$ and the proof is finished.

Examples. The following examples illustrate the various possibilities for F_x and also the fact that the upper bound in Theorem 4 is best possible.

Consider a (planar) half-disc in $\mathbb{R}^3$, take for a small $\varepsilon > 0$ its inner parallel convex set D_ε at distance $\pi\varepsilon$, and then the outer parallel convex body (in $\mathbb{R}^3$) of D_ε at distance 2ε.

We obtained a C^1 convex surface S containing a point x (which corresponds to the centre of the initial half-disc) and including a portion P isometric to a piece of a torus, so that F_x is almost half the largest circle on the torus. For $\varepsilon \to 0+$, we have $\mu_1(A \cup A') \to 2\pi$ and $\mu_1 F_x - \pi r_x \to 0$. For fixed ε and $r \to r_x-$, $\mu_1 C(x, r) \to 2\mu_1 F_x$.

If we take the longest circular arc $C_\varepsilon \subset D_\varepsilon$ and an arbitrary compact subset C'_ε of C_ε including the endpoints of C_ε, and then replace in the above construction D_ε by the convex hull of C'_ε, we get a set F_x congruent to $(1 + 2\varepsilon)\, C'_\varepsilon$.

A similar but nonsmooth example in which $\mu_1(A \cup A') = 2\pi$ can be obtained as follows. Consider the convex hull of a torus. Cut it along a plane of symmetry orthogonal to the plane Π of its largest circle. Cut again one of the two resulting pieces along Π and obtain two pieces of convex surface, bounded by closed Jordan curves having a common circular arc A on Π. By Aleksandrov's gluing theorem (see [1], p. 315–320), these pieces can be glued together along the Jordan curve (isometrically) keeping A as a common arc, to form a closed convex surface with the desired properties.

Further facts on farthest points

Also topologically F_x does not behave like the other circles $C(x, r)$. So a component of F_x may never look like the digit **8** or like the letter **A**, **O** or **P**, while $C(x, r)$ with smaller r may well do so. For very small r, $C(x, r)$ must be a closed Jordan curve, while F_x can never be that round. The following result presents another aspect of the answer to STEINHAUS' question.

Theorem 5. *For any point $x \in S$, every component of F_x is either a point or a Jordan arc.*

Proof. Suppose that the component K of F_x has more than a single point. Then it has at least two non-cutpoints a, b (see [11], p. 54). Clearly a and b are not isolated points, so we get as in the preceding proof the (not necessarily distinct) segments Σ'_a, Σ''_a joining a with x and the (not necessarily distinct) segments Σ'_b, Σ''_b joining b with x (see Figure 4, where $\Sigma'_a = \Sigma''_a$ and $\Sigma'_b \neq \Sigma''_b$).

Let D be the domain with boundary

$$\Sigma'_a \cup \Sigma''_a \cup \Sigma'_b \cup \Sigma''_b .$$

Since a and b are not cutpoints,

$$K \setminus \{a, b\} \subset D .$$

Let now $c \in K \setminus \{a, b\}$. There are points a', $b' \in K$ close enough to a, respectively b, to ensure that a', $b' \in C_x$ and take them so that, if $\Sigma'_{a'}$, $\Sigma''_{a'}$ are the segments from a' to x and $\Sigma'_{b'}$, $\Sigma''_{b'}$ those from b' to x (met in the order $\Sigma'_{a'}$, $\Sigma'_{b'}$, $\Sigma''_{b'}$, $\Sigma''_{a'}$ around x), then

$$\Sigma'_{a'} \cup \Sigma''_{a'} \cup \Sigma'_{b'} \cup \Sigma''_{b'}$$

separates c from a and from b.

Consider now the arc $A \subset C_x$ joining a' with b', found in the proof of Theorem 1, and let $u \in A \setminus \{a', b'\}$. The union of two of the segments from u to x separates a' from b' (see the mentioned proof). So, if $u \notin F_x$, then a' and b' lie in different components of F_x, a contradiction. Hence $A \subset K$. Suppose that for some point $u \in A \setminus \{a', b'\}$ there is a third segment from u to x besides the two mentioned above. Then the two angles (out of three) determined by these three segments at u towards a' and b' measure π each, because u is a limit point of sequences of points in F_x lying in the domains bounded by the corresponding pairs of segments; thus nothing remains for the third angle. This contradiction shows that each point $u \in A \setminus \{a', b'\}$ is joined with x by exactly two segments Σ'_u, Σ''_u, and choose the notation so that $\Sigma'_{a'}$, Σ'_u, $\Sigma'_{b'}$, $\Sigma''_{b'}$, Σ''_u, $\Sigma''_{a'}$ are met in this order around x. Then

$$\bigcup_{u \in A} (\Sigma'_u \cup \Sigma''_u) = \overline{D} .$$

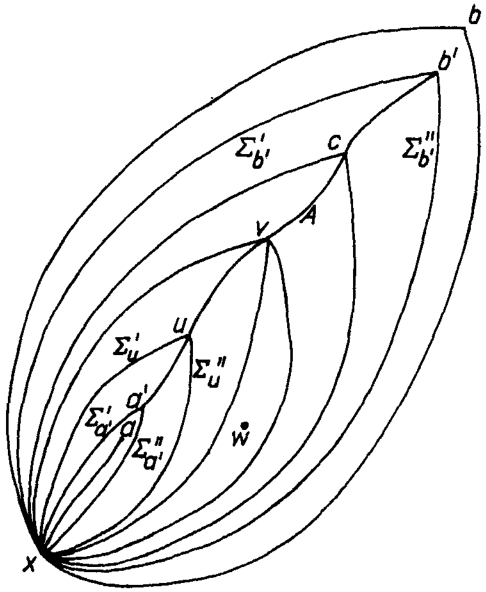

Fig. 4

Indeed, the existence of a point w in a component D' of $D \setminus A$, not belonging to $\Sigma'_u \cup \Sigma''_u$ for any $u \in A$ would imply the existence of a point $v \in A$ joined with x by two distinct segments through D' enclosing w between them; but then there were three segments from v to x, a contradiction.

Since no interior point of a segment Σ'_u or Σ''_u belongs to F_x, the chosen point c must lie on A. Moreover,

$$(\Sigma'_c \cup \Sigma''_c) \cap F_x = \{c\}$$

and $\Sigma'_c \cup \Sigma''_c$ separates a from b. Therefore c is a cutpoint of K. Hence K is a Jordan arc (see [11], p. 54).

Weak Wiedersehensflächen

Let $S \subset \mathbb{R}^3$ be a convex surface and $x \in S$. Let $g: [0, l] \to S$ describe a geodesic G starting at $g(0) = x$, with the arc-length as parameter, i.e., with s equal to the distance on G from x to $g(s)$. Let $z \in S$ and

$$d_{G,z}: G \to \mathbb{R}_+$$

be defined by $d_{G,z}(y) = \varrho(y, z)$. Suppose that, for some $z \in S$, $d_{G,z} \circ g$ is non-increasing in a connected neighbourhood of 0 and let $[0, \alpha]$ be a maximal such neighbourhood (by inclusion). Then we call $g(\alpha)$ a first proximum of G from z.

We call a weak Wiedersehensfläche any convex surface $S \subset \mathbb{R}^3$ such that for any point $x \in S$ there is some point $z \in S$ for which every geodesic starting in x has a first proximum from z precisely in z. In other words, going from any point $x \in S$ along a geodesic we eventually reach another point of S depending on x but not on the chosen geodesic, such that the distance to that point never increases.

We also recall that the specific curvature of a domain $D \subset S$ is $\omega(D)/\mu_2 D$, where $\omega(D)$ is the curvature of D (see [1], p. 418). The surface S is said to have bounded specific curvature if the specific curvatures of all domains in S have a finite upper bound.

We shall prove a statement lying in between HILBERT and COHN-VOSSEN's conjecture and the (established) Wiedersehensfläche conjecture of BLASCHKE, namely that every C^3 weak Wiedersehensfläche is a sphere.

Theorem 6. *Every weak Wiedersehensfläche with bounded specific curvature is a Wiedersehensfläche.*

Proof. Let S be a weak Wiedersehensfläche, $x \in S$, and z be the point from the above definition, common to all geodesics starting at x. Since S has bounded specific curvature, there is some $r_0 > 0$ so that for every tangent direction τ at z there is a segment of length r_0 starting at z in direction τ (see [1], p. 420).

Let G be a geodesic starting in x (see Figure 5). It contains z and let l be its length from x to z. Then the associated function $d_{G,z} \circ g$ is non-increasing on $[0, l]$.

Suppose $l > \varrho(x, z)$. Take a segment $Z \subset G$ starting in z, of length less than r_0. Consider the segment Σ_u from x to $u \in Z$. If $\Sigma_u \subset G$ for every $u \in Z$, then G is a segment and $l = \varrho(x, z)$, a contradiction. Hence $\Sigma_v \cap G = \{x, v\}$ for some $v \in Z$. The segment Σ_v can be extended beyond v to a geodesic H with parametrization h, crossing G and ending at z so that $d_{H,z} \circ h$ is non-increasing. Clearly G and H have distinct tangent directions at z. Let Z' be the segment of length r_0 starting at z in the same direction as H. Then $Z' \subset H$. We have

$$d_{H,z}(v) = d_{G,z}(v) < r_0 = d_{H,z}(z'),$$

where z' is the endpoint of Z' different from z. Hence $v \neq z'$. But, Z and Z' being different segments starting at z, necessarily $v \notin Z'$, so that the monotonicity hypothesis on $d_{H,z} \circ h$ is violated at v, z'.

Hence $l = \varrho(x, z)$, and S is a Wiedersehensfläche.

Corollary. *Each C^3 weak Wiedersehensfläche is a sphere.*

This follows from Theorem 6 combined with GREEN's result in [7].

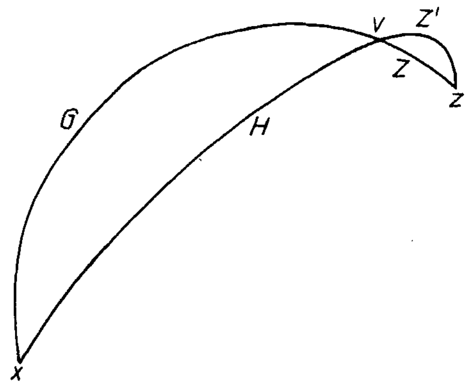

Fig. 5

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