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Most homeomorphisms of the circle are semiperiodic

By

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Several generic results about fixed points have been established by Vidossich [11], De Blasi [3], [4], Myjak [10], De Blasi and Myjak [5], Butler [1], Dominguez Benavides [7], and other authors. Recently, the last author of this paper showed that under the conditions of Schauder's theorem the set of fixed points is generically homeomorphic to the Cantor set (see [13]). The homeomorphisms of the sphere S^n may have no fixed points, even generically. The situation changes, however, if we restrict ourselves to homeomorphisms which are orientation preserving, for even dimension, or orientation reversing, for odd dimension. Our first author showed that, starting with dimension 2, such homeomorphisms have generically uncountably many fixed points (see [2]). On the circle S^1 each orientation reversing homeomorphism has precisely two fixed points, while the orientation preserving homeomorphisms may even have no periodic points!

It is easily seen that the space of all homeomorphisms of S^1 with the usual metric d is a Baire space, i.e. every open subset is of second Baire category. This is true, as well, for its subspace $\mathscr{H}$ of all orientation preserving homeomorphisms of S^1 and for the complementary subspace of all orientation reversing homeomorphisms.

In this paper we generically investigate the set P_h of all periodic points of the orientation preserving homeomorphisms h of S^1. We show that all these homeomorphisms except those in a nowhere dense set have periodic points and, for most $h \in \mathscr{H}$, the set P_h is homeomorphic to the Cantor set and $\lambda P_h = 0$, where λ denotes Lebesgue measure on S^1 and "most" means "all, except those in a first category set".

In $\mathbb{R}$ or in S^1, which is taken in the complex plane, let $\bar{B}(x, r)$ denote the (compact) arc of midpoint x and length $2r$; by $B(x, r)$ we denote its relative interior.

We shall use the notion of porosity, introduced by E. Dolzhenko [6], but essentially already known to Denjoy. We call a set $M \subset S^1$ *porous at* $x \in S^1$ if there is a positive number α such that for any $\varepsilon > 0$, there is a point $y \in B(x, \varepsilon)$ such that

$$B(y, \alpha|x - y|) \cap M = \emptyset$$

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(see [12]). If α can be chosen as close to 1 as we wish, M is called *strongly porous at* x. A set which is strongly porous at all points of S^1 is said to be *strongly porous*. Every set which is porous everywhere must have measure 0 by Lebesgue's density theorem.

If $h: S^1 \to S^1$ is an orientation preserving homeomorphism then one of the following cases occurs (see [8]).

I. All points of S^1 are periodic (and h is called *periodic*).
II. S^1 contains periodic and non-periodic points (h is *semiperiodic*).
III. No point of S^1 is periodic; there is no point x such that the set of points $\{x, h(x), h^2(x), \dots\}$ is everywhere dense on S^1 (h is *intransitive*).
IV. No point of S^1 is periodic; for some point x, the set of points $\{x, h(x), h^2(x), \dots\}$ is everywhere dense on S^1 (h is *transitive*).

Which case appears most frequently? We already anticipated the short answer: case II; this follows from the following result.

Theorem 1. *All orientation preserving homeomorphisms of S^1 except those in a nowhere dense set are semiperiodic.*

Proof. Consider first a homeomorphism $h: S^1 \to S^1$ of type I. Then all points have the same period n (see [8], p. 146), so h is topologically equivalent to a rotation R_θ of angle θ with rational θ/π, i.e. $h = g \circ R_\theta \circ g^{-1}$ with $g \in \mathscr{H}$. Then a rotation R_ϕ with irrational ϕ/π and with ϕ sufficiently close to θ yields a homeomorphism $g \circ R_\phi \circ g^{-1}$ of type IV arbitrarily close to h.

Since the set of all orientation preserving C^2-diffeomorphisms is dense in $\mathscr{H}$, for every $h \in \mathscr{H}$ and $\varepsilon > 0$ it is possible to find a C^2-diffeomorphism $f \in \mathscr{H}$ such that $d(f, h) < \varepsilon$, and f is not of type III (see [9], p. 149).

Hence the homeomorphisms of type II or IV lie densely in $\mathscr{H}$.

Consider now a homeomorphism h of type II or IV. If it is of type IV, it is topologically equivalent to a rotation R_θ of angle θ with θ/π irrational (see [8], p. 147), i.e. $h = g \circ R_\theta \circ g^{-1}$ with $g \in \mathscr{H}$. Because g and g^{-1} are uniformly continuous, for every $\varepsilon > 0$ there exists $\eta > 0$ such that $|y - y'| < \eta$ implies $|g(y) - g(y')| < \varepsilon$ and we can then find $\delta > 0$ such that $|z - z'| < \delta$ yields $|g^{-1}(z) - g^{-1}(z')| < \eta$. But for every natural number n the rotation $R_{n\theta}$ is an isometry, whence, for every $\varepsilon > 0$, we could find $\delta > 0$ such that, for all n, $|z - z'| < \delta$ implies

$$|h^n(z) - h^n(z')| = |g(R_{n\theta}(g^{-1}(z))) - g(R_{n\theta}(g^{-1}(z')))| < \varepsilon.$$

Hence, for any $\varepsilon > 0$, we can choose an arc $V \subset S^1$ and a natural number n such that $\lambda(V \cap h^n(V)) > 0$ and $\lambda(V \cup h^n(V)) < \varepsilon/3$. This is, of course, easily done in case h is of type II. We shall prove that it is possible to find a nonempty open set in $\mathscr{H}$ consisting of homeomorphisms of type II only, included in $\mathscr{B}(h, \varepsilon)$, the open ball of radius ε around h in $\mathscr{H}$.

Consider an arc $I \subset S^1$ and its middle third J. We define a homeomorphism $c: S^1 \to S^1$, which we call a *local contraction* associated to I, so that

i) c acts on J as a metrically linear contraction, leaving one of the points of J fixed,
ii) c is the identity outside I.

We can choose I so that $\lambda I < \varepsilon$ and $V \cup h^n(V) \subset J$. Let c be a local contraction associated to I with the additional property that $c(J) \subset \operatorname{int} K$, where $K = V \cap h^n(V)$ (the freedom in choosing the contraction in condition i) above allows us to do this).

Then $(c \circ h)(K) \subset h(K)$ and $(c \circ h)^n(K) \subset c \circ h^n(K) \subset c(J) \subset K$, so by Brouwer's fixed point theorem $(c \circ h)^n$ has a fixed point, that is $c \circ h$ has a periodic point. But $c(J) \subset \operatorname{int} K$, so there exists a $\delta > 0$ such that every $h' \in \mathscr{B}(c \circ h, \delta)$ has a periodic point.

We conclude that $\mathscr{B}(c \circ h, \delta)$ entirely consists of homeomorphisms admitting periodic points.

Actually this open ball contains only homeomorphisms of type II. Indeed, we saw at the beginning of the proof that arbitrarily close to every homeomorphism of type I there is a homeomorphism of type IV, which cannot belong to $\mathscr{B}(c \circ h, \delta)$.

This completes the proof of the theorem.

Theorem 2. *Most orientation preserving homeomorphisms of S^1 have a strongly porous set of periodic points.*

Proof. Let $M \subset S^1$. If, for every natural number n,

$$\forall x \in S^1, \quad \forall \varepsilon > 0, \quad \exists \eta \in (0, \varepsilon], \quad \text{such that } B(x, \eta) \cap M \subset B(x, \eta/n),$$

then the set M is strongly porous.

Let $\mathscr{H}_n$ be the family of all homeomorphisms h verifying the condition

$$\forall x \in S^1, \quad \exists \eta \in (0, 1/n], \quad \text{such that } B(x, \eta) \cap P_h \subset B(x, \eta/n).$$

If $f \in \mathscr{H}_n$ for all n, then P_f is strongly porous. We show that the complement of $\mathscr{H}_n$ in $\mathscr{H}$ is nowhere dense for any n.

Let $\mathscr{O}$ be an open set in $\mathscr{H}$. By Theorem 1, there is a semiperiodic function $f \in \mathscr{O}$. Let $\delta > 0$ be such that $d(f, g) < \delta$ implies $g \in \mathscr{O}$. Since f is uniformly continuous, we find an $\varepsilon > 0$, such that $d(x, y) < \varepsilon$ yields $|f(x) - f(y)| < \delta$. From the open covering $\{B(x, \varepsilon/2) : x \in P_h\}$ of the compact set P_h we select the finite subcovering $\{B(x, \varepsilon/2) : x \in F\}$. For every pair of consecutive points $x_i, x_{i+1} \in F$ at distance $d(x_i, x_{i+1}) > \varepsilon$, consider the largest open arc $y_i y_{i+1} \subset x_i x_{i+1} \setminus P_h$ and add y_i, y_{i+1} to F to get a new set F^*. Now for every pair of consecutive points in F^* there is no periodic point between them or their distance is less than ε. In the second case we replace the restriction of h to the arc between the two points by a strictly convex monotone function. Then it is easily seen that the function h^* constructed in this way has no periodic points outside F^*. Put

$$\Delta = \min \left\{1, \tfrac{1}{2} \min_{\substack{x, y \in F^* \\ x \neq y}} |x - y| \right\}$$

and

$$\xi = \min \left\{ |x - h^p(x)| : x \in S^1 \setminus \bigcup_{x \in F^*} B(x, \Delta/(2n)) \right\}.$$

For some $\nu > 0$, $d(g, h) < \nu$ implies $d(g^p, h^p) < \xi$, whence every fixed point of g^p must lie in some arc $B(x, \Delta/(2n))$ $(x \in F^*)$. Then g has the property that, in any arc of length 2Δ centred at a point y belonging to $B(x, \Delta/(2n))$ for some $x \in F^*$, the only periodic points lie in $B(x, \Delta/(2n)) \subset B(y, \Delta/n)$. Thus $g \in \mathscr{H}_n$ (if $d(g, h) < \nu$) and $\mathscr{H} \setminus \mathscr{H}_n$ is nowhere dense, as claimed. Hence most elements of $\mathscr{H}$ lie in $\bigcap_{n=1}^{\infty} \mathscr{H}_n$ and have therefore a strongly porous set of periodic points.

Lemma 1. *Let $I \subset \mathbb{R}$ be an open interval and $g: I \to \mathbb{R}$ an increasing continuous function. Let $x \in I$ and $\delta > 0$ be such that $\bar{B}(x, \delta) \subset I$. Then there is some $\varepsilon_0 \in (0, \delta)$ such that for every $\varepsilon \in (0, \varepsilon_0)$ there is some increasing continuous function $\bar{g}: I \to \mathbb{R}$ satisfying the properties*

i) *$\bar{g}(t) = g(t)$ for any $t \in I \setminus B(x, \delta)$*

ii) *for $y = x - 3\varepsilon$, $z = x + 3\varepsilon$, we have $\bar{g}(B(y, 2\varepsilon)) \subset B(\bar{g}(y), \varepsilon)$, $\bar{g}(B(z, 2\varepsilon)) \subset B(\bar{g}(z), \varepsilon)$, $\bar{g}(y) = g(y) - 3\varepsilon$, $\bar{g}(z) = g(z) + 3\varepsilon$.*

Proof. Let $\varepsilon_0 \in (0, \frac{\delta}{5})$ be such that $\bar{B}(g(x), 4\varepsilon_0) \subset g(B(x, \delta))$. For every fixed $\varepsilon \in (0, \varepsilon_0)$, define $\bar{g}$ as

$$\bar{g}(t) = \begin{cases} g(t), & \text{if } t \leqq x - \delta, \quad t \in I \\ g(x-\delta) + \dfrac{t - x + \delta}{\delta - 5\varepsilon}(g(x) - 4\varepsilon - g(x - \delta)), & \text{if } x - \delta \leqq t \leqq x - 5\varepsilon \\ g(x) - 4\varepsilon + \frac{1}{2}(t - x - 5\varepsilon), & \text{if } x - 5\varepsilon \leqq t \leqq x - \varepsilon \\ g(x) + 2(t - x), & \text{if } x - \varepsilon \leqq t \leqq x + \varepsilon \\ g(x) + 2\varepsilon + \frac{1}{2}(t - x - \varepsilon), & \text{if } x + \varepsilon \leqq t \leqq x + 5\varepsilon \\ g(x) + 4\varepsilon + \dfrac{t - x - 5\varepsilon}{\delta - 5\varepsilon}(g(x - \delta) - g(x) - 4\varepsilon), & \text{if } x + 5\varepsilon \leqq t \leqq x + \delta \\ g(t), & \text{if } x + \delta \leqq t, \ t \in I. \end{cases}$$

From the manner ε_0 was chosen it follows that

$$g(x - \delta) < g(x) - 4\varepsilon, \quad g(x) + 4\varepsilon < g(x + \delta).$$

Therefore $\bar{g}$ is well defined, increasing, continuous, and satisfies i) and ii).

Lemma 2. *Let $h \in \mathscr{H}_p$ be semiperiodic and $x^0 \in P_h$, $x^i = h(x^{i-1})$ $(1 \leqq i \leqq p)$. Let $\delta > 0$ be such that the arcs $\bar{B}(x^i, \delta)$ are mutually disjoint for $0 \leqq i \leqq p - 1$. Then there is some $\varepsilon_0 \in (0, \frac{\delta}{5})$ such that for every $\varepsilon \in (0, \varepsilon_0)$ there is some $\bar{h} \in \mathscr{H}_p$ satisfying the properties*

i) *$\bar{h}(x) = h(x)$ for any $x \in S^1 \setminus \bigcup_{i=0}^{p-1} \bar{B}(x^i, \delta)$*

ii) *if $y^i z^i = \bar{B}(x^i, 3\varepsilon)$ then $\bar{h}(B(y^i, 2\varepsilon)) \subset B(y^{i+1}, \varepsilon)$ and $\bar{h}(B(z^i, 2\varepsilon)) \subset B(z^{i+1}, \varepsilon)$, where $y^{i+1} = h(y^i)$ and $z^{i+1} = h(z^i)$ $(0 \leqq i \leqq p - 1)$.*

Proof. Let $G_i \subset S^1$ be disjoint connected open sets such that $\bar{B}(x^i, \delta) \subset G_i$ for $0 \leqq i \leqq p - 1$. For any G_i consider an open interval $I_i \subset \mathbb{R}$ such that G_i can be identified with I_i, the distance and the orientation remaining unchanged. The same can be done for $h(G_i)$. Thus we obtain from $h|_{G_i}$ an increasing continuous function $h_i: I_i \to \mathbb{R}$. Using Lemma 1 we obtain for h_i a number ε_0^i. Apply this procedure for every i and define $\varepsilon_0 = \min\{\varepsilon_0^i : 0 \leqq i \leqq p - 1\}$.

Let $\varepsilon \in (0, \varepsilon_0)$. We can now define $\bar{h}(x)$ as $h(x)$ for any $x \in S^1 \setminus \bigcup_{i=0}^{p-1} G_i$; for $x \in G_i$ we define $\bar{h}(x)$ as the image of $\bar{h}_i(x)$ through the above identification. According to Lemma 2, $\bar{h}$ verifies the required properties.

Theorem 3. *For most orientation preserving homeomorphisms of S^1, the set of periodic points is homeomorphic to the Cantor set.*

Proof. According to Theorem 2, for most elements $h \in \mathscr{H}$ the set P_h is totally disconnected. It will be then sufficient to prove that P_h is perfect for most h. Denote

$$\mathscr{F}_n = \{h \in \mathscr{H} : |x - y| \geqq \tfrac{1}{n} \quad \text{for some } x \in P_h \quad \text{and any } y \in P_h \setminus \{x\}\}.$$

Let $h \in \mathscr{F}_n$ be of period p. Let $\{x_1, \ldots, x_m\}$ be a $\frac{1}{2n}$-net of P_h, and $\eta > 0$ a real number. Denote

$$G = S^1 \setminus \bigcup_{j=1}^{m} B\left(x_j, \frac{1}{2n}\right).$$

The set G is compact. Notice that the function $h^p = h \circ \cdots \circ h$ (p times) cannot have fixed points in G, because

$$P_h \subset \bigcup_{j=1}^{m} B\left(x_j, \frac{1}{2n}\right).$$

Therefore $\min\limits_{x \in G} |h^p(x) - x| > 0$.

Let $\alpha > 0$ be such that $d(g, h) < \alpha$ implies

$$\min_{x \in G} |g^p(x) - x| > 0.$$

Set $x_j^0 = x_j$, $x_j^i = h(x_j^{i-1})$ for $1 \leqq j \leqq m$, $1 \leqq i \leqq p$. Denote

$$\delta = \min\{|x_j^i - x_l^k| : x_j^i \neq x_l^k, \quad 1 \leqq i, k \leqq p;\ 1 \leqq j, l \leqq m\}.$$

Let $\delta_0 \in (0, \delta)$ be such that

$$\operatorname{diam} h(B(x_j^i, \delta_0)) < \min\left\{\eta, \frac{\alpha}{2}\right\}.$$

Denote by ε_0^j the positive number obtained by applying the previous Lemma to the function h, point x_j, and number $\delta = \delta_0$, where $1 \leqq j \leqq m$.

Now choose $\varepsilon \in (0, \min\{\varepsilon_0^j : 1 \leqq j \leqq m\})$ such that $\varepsilon < \min\{\frac{\alpha}{2}, \frac{1}{10n}\}$. Apply Lemma 2 to x_1, the function h and this ε, and denote by h_1 the resulting function. Then apply the same Lemma to x_2, the function h_1, and ε, and denote the resulting function by h_2, and so on. Finally, for x_m, h_{m-1}, and ε we obtain the function h_m.

Thus, the function $\bar{h} = h_m \in \mathscr{H}$ satisfies

i) $\bar{h}(x) = h(x)$, for any $x \in S^1 \setminus \bigcup \{B(x_j^i, \delta_0) : 1 \leqq i \leqq m, 1 \leqq j \leqq p - 1\}$

ii) if $y_j^i z_j^i = \bar{B}(x_j^i, 3\varepsilon)$ then $\bar{h}(B(y_j^{i-1}, 2\varepsilon)) \subset B(y_j^i, \varepsilon)$ and $\bar{h}(B(z_j^{i-1}, \varepsilon)) \subset B(z_j^i, \varepsilon)$, for any $1 \leqq j \leqq m$, $1 \leqq i \leqq p$.

Let us notice that for any $x \in S^1$ such that $\bar{h}(x) \neq h(x)$ there is a unique point x_j^i with $x \in B(x_j^i, \delta_0)$; we also have

$$\bar{h}(B(x_j^i, \delta_0)) = h(B(x_j^i, \delta_0)), \quad \operatorname{diam} h(B(x_j^i, \delta_0)) < \eta.$$

Therefore $d(h, \bar{h}) < \eta$. We shall prove now that

$$B(\bar{h}, \varepsilon) \cap \mathscr{F}_n = \emptyset.$$

Let $g \in B(\bar{h}, \varepsilon)$. Then

$$g(B(y_j^{i-1}, 2\varepsilon)) \subset B(y_j^i, 2\varepsilon), \quad g(B(z_j^{i-1}, 2\varepsilon)) \subset B(z_j^i, 2\varepsilon).$$

Therefore

$$g^p(B(y_j^0, 2\varepsilon)) \subset B(y_j^0, 2\varepsilon)), \quad g^p(B(z_j^0, 2\varepsilon)) \subset B(z_j^0, 2\varepsilon)) \quad (1 \leqq j \leqq m).$$

Hence there are some fixed points $\bar{y}_j^0 \in \bar{B}(y_j^0, 2\varepsilon)$, $\bar{z}_j^0 \in \bar{B}(z_j^0, 2\varepsilon)$ of g^p, i.e. $\bar{y}_j^0, \bar{z}_j^0 \in P_g$. On the other hand

$$d(g, h) \leqq d(g, \bar{h}) + d(\bar{h}, h) < \varepsilon + \min\left\{\eta, \frac{\alpha}{2}\right\} \leqq \alpha,$$

so that $\min_{x \in G} |g^p(x) - x| > 0$. Then

$$P_g \subset S^1 \setminus G = \bigcup_{j=1}^{m} B\left(x_j, \frac{1}{2n}\right).$$

Let $w \in P_g$. First, suppose that $w \neq \bar{y}_j^0, \bar{z}_j^0$ $(1 \leqq j \leqq m)$. Then there is some j such that $w \in B(x_j, \frac{1}{2n})$. For this j we obtain

$$|w - \bar{y}_j^0| \leqq |w - x_j| + |x_j - \bar{y}_j^0| < \frac{1}{2n} + 5\varepsilon < \frac{1}{n}.$$

Suppose now that $w = \bar{y}_j^0$ for some j. Then

$$|w - \bar{z}_j^0| \leqq 10\varepsilon < \frac{1}{n}.$$

Therefore $g \notin \mathscr{F}_n$.

We have proved that for any $h \in \mathscr{F}_n$ and $\eta > 0$ there is some $\bar{h} \in \mathscr{H}$ such that $d(h, \bar{h}) < \eta$ and

$$B(\bar{h}, \varepsilon) \cap \mathscr{F}_n = \emptyset.$$

This means that $\mathscr{F}_n$ is nowhere dense in $\mathscr{H}$. Therefore $\mathscr{H} \setminus \bigcup_{n \geqq 1} \mathscr{F}_n$ is residual in $\mathscr{H}$, hence most elements of $\mathscr{H}$ have a perfect set of periodic points.

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