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A characterization of infinite, bipartite Toeplitz graphs

by

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Abstract

Toeplitz graphs are graphs whose adjacency matrix is Toeplitz. Infinite Toeplitz graphs having no loops and no multiple edges can thus be defined by an infinite 0-1 sequence. We characterize bipartite such Toeplitz graphs by introducing the concept of odd T-cycles and by using the base-circuit language of matroid theory.

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1) Introduction

An $n\times n$ matrix $A=(a_{ij})$ is called Toeplitz, if $a_{ij}=a_{(i+1)(j+1)}$ for $i,j=1,\ldots,n$. Since any diagonal of such a matrix contains identical elements it is uniquely determined by its first row and column and therefore easy to memorize. Beyond this straightforward advantage there are other particular applications of these matrices in computer science (cf. Aho et al. [1, p.249], JaJa [7]) and also in mathematics or physics.

A Toeplitz graph is a graph whose adjacency matrix is Toeplitz. Not much is known to date about structural and algorithmic properties of these graphs. Connectivity properties have been studied by Garfinkel et al. [6] , Burkard and Sandholzer [3], van Doorn [5], and hamiltonian properties by Medova and Dempster [8], van der Veen et al. [9] and van Dal et al. [4].

The Toeplitz graphs we consider in this paper are infinite, symmetric ,without loops and multiple edges; they will be denoted by $T=(\mathbb{N}, E)$. Consequently, any such graph is uniquely defined by a sequence from $\{0,1\}^{\mathbb{N}}$, whose first element is 0 (throughout the paper we assume that this is the case for the sequences we study.). Given two such sequences A and B we say that A dominates B (or $A\geq B$) if $A_i\geq B_i$ for all $i\in\mathbb{N}$. We will make heavy use of the concept of an odd T-cycle, i.e. a sequence from $\{0,1\}^{\mathbb{N}}$, which induces ≥ 1 odd cycles in its associated Toeplitz graph and whose number of 1-entries is minimal with respect to this property. Our main result consists in characterizing the bipartite graphs among the Toeplitz graphs under study by the non-domination of odd T-cycles. We also fully describe the maximal bipartite such graphs. We think that our structural results might help to also obtain algorithmic results, for instance in relation with the max-cut problem over such graphs.

It will become clear from the structure of odd T-cycles that the finite case cannot be treated the same way but instead requires some special attention. This situation and conditions for planarity of Toeplitz-graphs are the subject of a companion paper.

2) Bipartite Toeplitz graphs

For $\alpha \in \mathbb{N}$ let $\underline{B}^\alpha$ denote the $\{0,1\}^{\mathbb{N}}$-sequence (consisting of α 0's first, then of a 1 etc.)

$$(\underbrace{0 \dots 0}_{\alpha}\ 1\ \underbrace{0 \dots 0}_{\alpha-1}\ \underbrace{0 \dots 0}_{\alpha}\ 1\ \underbrace{0 \dots 0}_{\alpha-1}\ \underbrace{0 \dots 0}_{\alpha}\ 1\ \underbrace{0 \dots 0}_{\alpha-1}\ \dots),$$

and let $\underline{T}^\alpha$ denote the associated Toeplitz graph. As illustrated in Figure 1, T^α is the disjoint union of complete, bipartite subgraphs. The nodes $\mathbb{N}$ of T^α can be classified according to the remainder r mod 2α, i.e.

i) $0 \le r < \alpha$ and ii) $\alpha \le r < 2\alpha$.

More precisely, a node $i \in \mathbb{N}$ is either

i') an even multiple of α plus the remainder r with $0 \le r < \alpha$,

or

ii') an odd multiple of α plus the remainder r with $0 \le r < \alpha$.

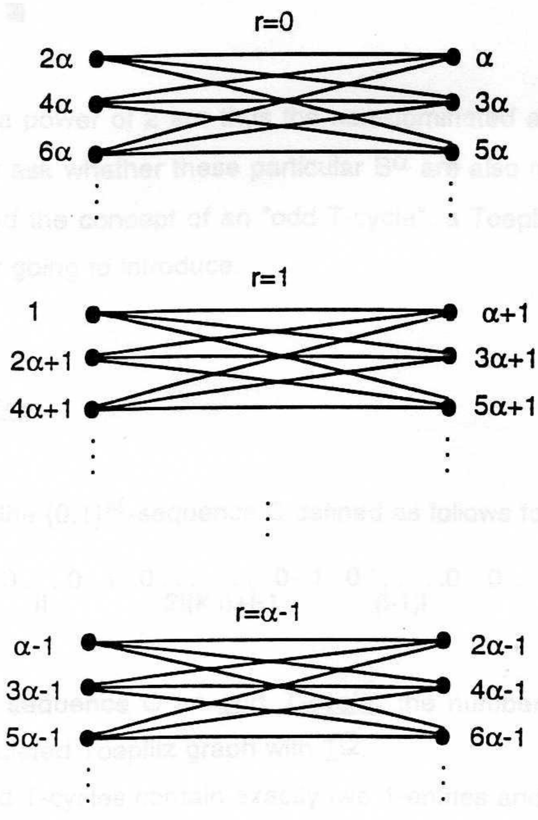


Figure 1

Observe that if $\alpha=1$ the graph T^{α} reduces to a (single) complete, bipartite graph over the nodes $2\mathbb{N}$ and $2\mathbb{N}-1$.

Proposition 1

If $\alpha \equiv \beta \bmod 2\beta$ then $B^{\alpha} \leq B^{\beta}$.

Proof:

Consider B^{α} and subdivide it according to the condition given. ■

Proposition 2

Let $\beta \in \mathbb{N}$, $\beta \neq 2^m$ for all $m \in \{0,1,2,...\}$. Then there is an α with $\alpha=2^{m'}$ for some $m' \in \{0,1,2,...\}$ such that $B^{\beta} \leq B^{\alpha}$.

Proof:

Obvious from the the fact that β contains an odd divisor and from the condition given in Proposition 1. ■

The B^{α} with α a power of 2 are thus the non-dominated among all these sequences. We may further ask whether these particular B^{α} are also maximal w.r.t. bipartiteness. For this we need the concept of an "odd T-cycle", a Toeplitz-analogon to odd cycles, that we are now going to introduce.

3) Odd T-cycles

Let us consider the $\{0,1\}^{\mathbb{N}}$-sequence C defined as follows for $l \in \mathbb{N}$, $k \in \mathbb{N}$ and $i \in \{1,...,k\}$:

$$C = (\underbrace{0 \ldots 0}_{il}\ 1\ \underbrace{0 \ldots\ldots 0}_{2l(k-i)+l-1}\ 1\ \underbrace{0 \ldots\ldots 0}_{(i-1)l}\ 0\ \ldots)\ .$$

We call such a sequence C an <u>odd T-cycle</u>, the number 2kl+1 its <u>length</u>, and we denote the associated Toeplitz graph with $\underline{T^{C}}$.

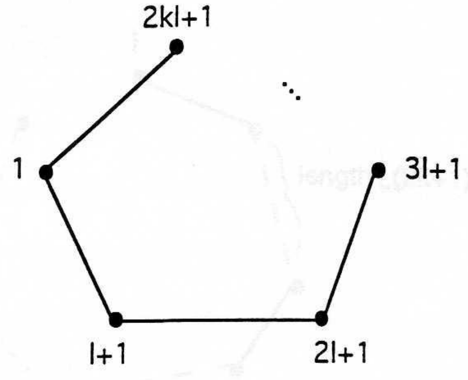
Observe that odd T-cycles contain exactly two 1-entries and ,thus, give rise to another (infinite) symmetric graph, whose study could be of interest.

Lemma 1

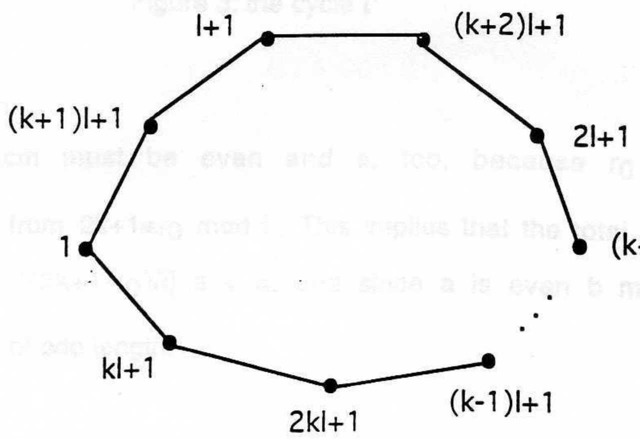
T^C contains an odd cycle.

Proof:

Let us first illustrate two simple instances:



a) the case i=1



b) the case i=k

Figure 2

c) the general case:

We suppose that $2k+1 \equiv r_0 \bmod i$.

c1) If $r_0=0$, the there is a cycle in T^C of total length (2k+1)/i , which is odd.

c2) For $r_0>0$ let <u>lcm</u> denote the least common multiple of r_0 and i and let lcm=ar_0=bi with a and b prime to each other. Then we can identify within T^C a cycle Γ of total length $[(2k+1-r_0)/i]$ lcm/r_0 + lcm/i, as shown in Fig.3:

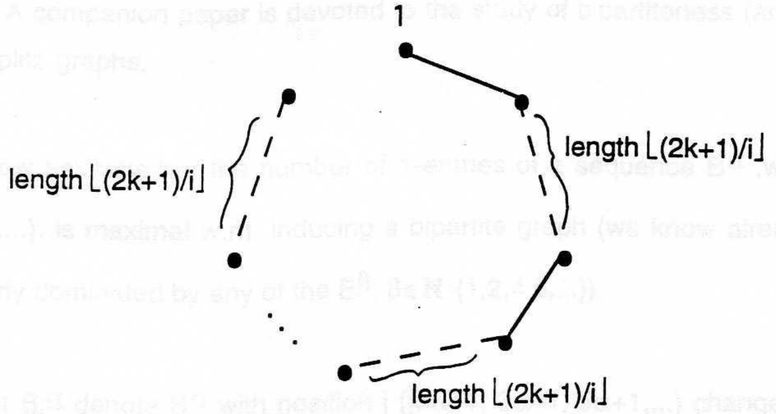


Figure 3: the cycle Γ

<u>Case 1</u>: i is even:

Then lcm must be even and a, too, because r_0 is odd as follows from $2k+1 \equiv r_0$ mod i . This implies that the total length of Γ equals $[(2k+1-r_0)/i]$ a + b, and since a is even b must be odd and Γ is of odd length.

<u>Case 2</u>: i is odd:

Subcase 2.1: lcm is also odd. Then b must be odd and so are a and r_0. The latter implies $(2k+1-r_0)/i$ to be even, and therefore the total length of Γ is odd.

Subcase 2.2: lcm is even. This implies b to be even and a, r_0 to be odd, even, respectively. Hence $(2k+1-r_0)/i$ is odd and, altogether, Γ is shown to be of odd length. ■

Observe that changing one of the two 1's to 0 within such an odd T-cycle C leads to a sequence inducing a bipartite Toeplitz graph. Therefore, C (i.e. its number of 1-entries) is minimal w.r.t. inducing an odd cycle in T^C. Also note that more than one such odd cycle can be induced in T^C. Finally, observe that to fully induce these odd cycles it is (in general) necessary that T^C contains at least 2kl+1 nodes. This is always the case for the node set $\mathbb{N}$. However, if the node set is finite, the situation needs special attention. A companion paper is devoted to the study of bipartiteness (and planarity) of finite Toeplitz graphs.

We will now ask whether the number of 1-entries of a sequence B^{α} ,with $\alpha=2^m$ and $m\in\{0,1,2,...\}$, is maximal w.r.t. inducing a bipartite graph (we know already that B^{α} is not properly dominated by any of the B^{β}, $\beta\in\mathbb{N}-\{1,2,4,8,...\}$).

For this let $\underline{B_j^{\alpha}}$ denote B^{α} with position j ($j\neq\alpha+1$, $3\alpha+1$, $5\alpha+1$,...) changed from 0 to 1. We distinguish 4 main cases:

<u>Case 1</u>: $\alpha=1$ i.e. $B_j^{\alpha} = (0\,1\,0\,1\,0\,1 \ldots 0\,1\,\overset{j}{1}\,1\,0\,1\,0 \ldots)$:

with $l=1$, $k=i=j-2$ there is an odd T-cycle dominated by B_j^{α}.

<u>Case 2</u>: $j\equiv 2p \bmod 2\alpha$, $0\leq p\leq \alpha-1$, $j\geq\alpha+2$:

with $l=1$, $i=\alpha$ and $k=(j+\alpha-2)/2$ there is an odd T-cycle dominated by B_j^{α}.

<u>Case 3</u>: $j\equiv 2p+1 \bmod 2\alpha$, $0\leq p\leq\alpha-1$, $j\geq\alpha+2$ and $p\neq\alpha/2$:

subcase 3.1: $p=0$ (observe that this case also covers the situation $\alpha=2$):

with $l=\alpha$, $i=1$ and $k=(j-1)/2\alpha$ there is an odd T-cycle dominated by B_j^{α}.

subcase 3.2: $1\leq p\leq\alpha-1$. Let $p=2^r t$ with t odd. Then with $l=2^{r+1}$, $i=2^{m-(r+1)}$ and $k=(t-1)/2+2^{m-r-2}(2\beta+1)$, where β comes from $j=(2p+1)+\beta\, 2\alpha$, there is an odd T-cycle dominated by B_j^{α}.

Case 4: $2 \le j \le \alpha$:

subcase 4.1: $\alpha - j = 2p$, $p \in \{0,1,2,\dots\}$:

with $l=1$, $i=j-1$ and $k=(\alpha+j-2)/2$ there is an odd T-cycle dominated by B_j^{α}.

subcase 4.2: $\alpha - j = 2p+1$, $p \in \{0,1,2,\dots\}$:

- if p is even then with $l=2$, $i=(j-1)/2$ and $k=(\alpha-p)/2$ we are done;
- if p is odd let $p+1=2^r t$, t odd. With $l=2^{r+1}$, $i=2^{m-(r+1)}$ and $k=(t-1)/2+2^{m-r-1}$ there is an odd T-cycle dominated by B_j^{α}.

This completes the proof of

Lemma 2

A sequence B^{α} with $\alpha \in \{1,2,4,\dots\}$ is maximal w.r.t. inducing a bipartite Toeplitz graph.

4) The main result

We will now proceed to show that the B^{α}, $\alpha=1,2,4,8,\dots$, are the *only* 0-1 sequences that induce bipartite Toeplitz graphs and that are not properly dominated by any other such sequence. For this let us consider a sequence $I \in \{0,1\}^{\mathbb{N}}$ which we suppose to induce a bipartite Toeplitz graph. By Lemma 1, I cannot dominate an odd T-cycle.

Suppose that I is not dominated by any of the B^{α}, $\alpha=1,2,4,8,\dots$. To obtain a contradiction we will exhibit an odd T-cycle dominated by I. Let j_1 and j_2 be the indices of the first and second 1-entry in I, respectively, and let $\gamma=j_1-1$, $\delta=j_2-j_1-1$, as indicated in Figure 4:

$$I = (\underbrace{0 \dots\dots 0}_{\gamma}\ \overset{j_1}{1}\ \underbrace{0 \dots\dots 0}_{\delta}\ \overset{j_2}{1}\ \dots\ ) .$$

Figure 4

Proposition 3

δ must be odd.

Proof:

Suppose $\delta=2p$, $p\in\{0,1,2,\ldots\}$. Then with $k=i+p$, $l=1$ and $i=\gamma$ we can exhibit an odd T-cycle dominated by I. ■

Consequently, for some $\alpha\in\mathbb{N}$, I can be written as

$$I = (\underbrace{0\ldots\ldots0}_{\gamma}\ \overset{j_1}{1}\ \underbrace{0\ldots\ldots0}_{\alpha-1}\ \underbrace{0\ldots\ldots0}_{\alpha}\ \overset{j_2}{1}\ \ldots\ ).$$

Figure 5

a) We will first show that for our I there exists β, $\beta=2^m$ with $m\in\{0,1,2,\ldots\}$ such that

$\alpha\equiv0 \bmod \beta$ and $\gamma\equiv\beta \bmod 2\beta$; in other words I has the following form:

$$I = (\underbrace{0.0}_{\beta}\ \underbrace{0..0}_{2\beta}\ \ldots\ \underbrace{0..0}_{2\beta}\ \overset{j_1}{1}\ \underbrace{0.0}_{\beta}\ \underbrace{0.0}_{\beta}\ \ldots\ \underbrace{0.0}_{\beta}\ \overset{j_2}{1}\ \ldots\ ),$$

Figure 6

So let us suppose that I is not of this form.

Proposition 4

If $\alpha=2^r t$ with t odd and if $\gamma\not\equiv\beta \bmod 2\beta$ for any divisor β of α then $\gamma\equiv0 \bmod 2^{r+1}$.

Proof:

t is odd and so are all of its divisors. It is therefore sufficient to assume that $\gamma \not\equiv \beta \bmod 2\beta$ for $\beta=1,2,...,2^r$. But then the only remaining case is $\gamma \equiv 0 \bmod 2^{r+1}$. ■

It is for this particular case that we are now going to exhibit an odd T-cycle dominated by I. Let $\gamma=2^{r+1}p$ for some $p \in \mathbb{N}$. Then we choose $l=2^{r+1}$, $i=p$ and $k=p+(t-1)/2$ to obtain

$$2l(k-i)+l-1 = 2^{r+1}(2(k-p)+1)-1 = 2^{r+1}t-1 = 2\alpha-1, \text{ as required.}$$

We can thus complete part a) with the conclusion that if I induces a bipartite Toeplitz graph then there is a β with $\beta \mid \alpha$ and $\gamma \equiv \beta \bmod 2\beta$. Moreover, β can be chosen to be a power of 2.

b) So let $\beta=2^m$ for some $m \in \{0,1,2,...\}$, let j_3 be the index of a third 1-entry in our sequence I; we set $\varepsilon=j_3-j_1-1$.

We assume that $\varepsilon \not\equiv 2^{m+1}-1 \bmod 2^{m+1}$ and that $\varepsilon \geq 2^{m+1}$. Then either $\varepsilon \equiv 0 \bmod 2$ or $\varepsilon \equiv 1 \bmod 4$ or ... or $\varepsilon \equiv 2^m-1 \bmod 2^{m+1}$. For any of these cases we will now exhibit an odd T-cycle which is dominated by I, the desired contradiction.

So let $\varepsilon=(2^q-1)+p2^{q+1}$ for $p \in \mathbb{N}$ and $q \in \{0,1,...,m\}$. We choose $l=2^q$, $i=2^{m-q}$ and $k=i+p$. Then we have

$$il = 2^m \quad \text{and} \quad 2l(k-i)+l-1 = 2^{q+1}p+2^q-1 = \varepsilon,$$

which yields an odd T-cycle dominated by I.

Altogether we have shown:

Theorem

An infinite 0-1 sequence I induces a bipartite Toeplitz graph

if and only if

I does not dominate an odd T-cycle

if and only if

I is dominated by one of the sequences B^{α}, where $\alpha \in \{1,2,4,8,...\}$.

5) Conclusion and final remarks

As already mentioned in the introduction, we feel that studying (infinite or finite) Toeplitz graphs might not only lead to structural results: several types of optimization problems such as the stable set problem, the matching problem or the max-cut problem might be faster solvable for these graphs.It could also be helpful within this context to study the facial structure of related polyhedra.

As for bipartiteness in finite Toeplitz graphs and results on planarity we refer the reader to our companion paper currently being written.

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