Hamiltonian properties of Toeplitz graphs

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Abstract

Conditions are given for the existence of hamiltonian paths and cycles in the so-called Toeplitz graphs, i.e. simple graphs with a symmetric Toeplitz adjacency matrix.

Keywords: Toeplitz graph; Hamiltonian graph; Traceable graph

0. Introduction[2]

An $(n \times n)$ matrix $A = (a_{ij})$ is called a *Toeplitz matrix* if $a_{ij} = a_{i+1,j+1}$ for each $i, j = 1, \ldots, n - 1$. Toeplitz matrices are precisely those matrices that are constant along all diagonals parallel to the main diagonal, and thus a Toeplitz matrix is determined by its first row and column. Toeplitz matrices occur in a large variety of areas in pure and applied mathematics. For example, they often appear when differential or integral equations are discretized, they arise in physical data-processing applications, and in the theories of orthogonal polynomials, stationary processes, and moment problems; see Heinig and Rost [9]. Other references on Toeplitz matrices are Gohberg [8] and Iohvidov [10].

$$T = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$
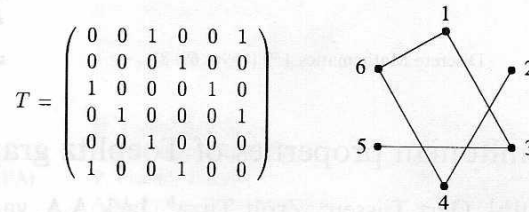
Fig. 1. A symmetric Toeplitz adjacency matrix T and the corresponding graph.

A *Toeplitz graph* is a (undirected) graph with a symmetric Toeplitz adjacency matrix. Therefore, an $(n \times n)$ matrix $B = (b_{ij})$ is the adjacency matrix of the Toeplitz graph G on n vertices if B is a 0–1 Toeplitz matrix, B is symmetric, and for all $i, j = 1, \dots, n$ the following holds: the edge $\{i, j\}$ is in the edge set of G if and only if $b_{ij} = b_{ji} = 1$. In this paper we describe hamiltonian properties of Toeplitz graphs. The n distinct diagonals of an $(n \times n)$ symmetric Toeplitz adjacency matrix will be labeled $0, 1, 2, \dots, n-1$. Diagonal 0 is the main diagonal and it contains only zeros, i.e. $a_{ii} = 0$ for all $i = 1, \dots, n$ so that there are no loops in the Toeplitz graph. Let $t_1, t_2, \dots, t_k$ be the diagonals containing ones ($0 < t_1 < t_2 < \cdots < t_k < n$). Then, the corresponding Toeplitz graph will be denoted by $T_n \langle t_1, \dots, t_k \rangle$. That is, $T_n \langle t_1, \dots, t_k \rangle$ is the graph with vertex set $1, 2, \dots, n$ in which the edge $\{i, j\}$, $1 \leqslant i < j \leqslant n$, occurs if and only if $j - i = t_l$ for some l, $1 \leqslant l \leqslant k$. For example, let $n = 6, k = 2, t_1 = 2$, and $t_2 = 5$. Fig. 1 shows the symmetric Toeplitz adjacency matrix T and the Toeplitz graph $T_6 \langle 2, 5 \rangle$.

Closely related to Toeplitz matrices are the so-called circulant matrices. An $(n \times n)$ matrix C is called a *circulant matrix* if it is of the form

$$\begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & & & & & \vdots \\ c_1 & c_2 & c_3 & \cdots & c_{n-1} & c_0 \end{pmatrix}.$$

For each $i, j = 1, \dots, n$ and $k = 0, 1, \dots, n-1$, all the elements (i, j) such that $j - i = k \,(\mathrm{mod}\, n)$ have the same value c_k; these elements form the so-called kth *stripe* of C. Obviously, a circulant matrix is determined by its first row (or column). It is clear that every circulant matrix is a Toeplitz matrix, but the converse is not necessarily true. Circulant matrices and their properties have been studied extensively in Davis [5].

Several authors have formulated conditions for connectivity and Hamiltonicity of circulant (di)graphs, i.e. (di)graphs with a circulant adjacency matrix. Garfinkel [7] proved that the number of dicycles associated with the kth stripe of an $(n \times n)$ circulant matrix is given by $\gcd(k, n)$. Boesch and Tindell [1] characterized the circulant graphs which are connected and conjectured that all connected circulant graphs are

hamiltonian. This conjecture has been proven by Burkard and Sandholzer [2]. Van Doorn [6] derived an explicit expression for the connectivity of circulant digraphs. Medova-Dempster [11] considered the asymmetric TSP (Traveling Salesman Problem) for circulant matrices and conjectured that this problem is $\mathcal{NP}$-hard in the general case. Finally, Van der Veen et al. [13] described two heuristics for the TSP restricted to symmetric circulant matrices and showed that these two heuristics are superior to some well-known heuristics for solving the general symmetric TSP.

Our motivation for considering hamiltonian properties of Toeplitz graphs is twofold. First, we wish to study these properties for graphs that have adjacency matrices from a broader class than that of the circulant adjacency matrices. Our second aim is to investigate which classes of subgraphs of the complete graph the TSP to be efficiently solvable (note that if the TSP for circulant matrices turns out to be $\mathcal{NP}$-hard, then the TSP for Toeplitz matrices is also $\mathcal{NP}$-hard). At present there are only few examples of such classes of subgraphs. Cornuéjols et al. [3, 4] gave a polynomial-time algorithm for the TSP restricted to Halin graphs and Ratliff and Rosenthal [12] solved efficiently the TSP for a graph that models a rectangular warehouse.

We are indebted to Gerard Sierksma for his moral support and for inviting Tudor Zamfirescu to visit the university of Groningen to participate in joint research. This co-operation resulted in the present paper.

1. Nonhamiltonian and nontraceable Toeplitz graphs

We start with a few simple results providing necessary conditions for a Toeplitz graph $T_n \langle t_1, \ldots, t_k \rangle$ to be hamiltonian or traceable, where a graph is called *traceable* if it admits a hamiltonian path. Clearly, a necessary condition for traceability is connectedness and a necessary condition for Hamiltonicity is 2-connectedness. The following theorem gives a lower bound on the number of components of a Toeplitz graph.

Theorem 1. $T_n \langle t_1, \ldots, t_k \rangle$ *has at least* $\gcd(t_1, \ldots, t_k)$ *components.*

Proof. It will be shown that the vertices $1, \ldots, \gcd(t_1, \ldots, t_k)$ are all in different components. Let $u, v \in \{1, \ldots, \gcd(t_1, \ldots, t_k)\}$ and $u \neq v$. Assume that the vertices u and v are in the same component, i.e. there is a path joining u and v. So, there are $\lambda_i \in \mathbb{Z}$, $i = 1, \ldots, k$ such that

$$u = v + \sum_{i=1}^{k} \lambda_i t_i.$$

Therefore, there is a $\lambda \in \mathbb{Z} \setminus \{0\}$ such that $u = v + \lambda \gcd(t_1, \ldots, t_k)$ which contradicts the assumption that $u, v \in \{1, \ldots, \gcd(t_1, \ldots, t_k)\}$ and $u \neq v$. $\square$

Corollary 1. *If* $\gcd\{t_1, \ldots, t_k\} > 1$, *then* $T_n \langle t_1, \ldots, t_k \rangle$ *is disconnected.*

Note that $T_n\langle t_1, \ldots, t_k \rangle$ can have more than $\gcd(t_1, \ldots, t_k)$ components. For instance, consider $T_5\langle 3, 4 \rangle$ which has 2 components whereas $\gcd(3, 4) = 1$. On the other hand, Burkard and Sandholzer [2] showed that if $T_n\langle t_1, \ldots, t_k \rangle$ is a circulant graph, i.e. if t_l occurs then $n - t_l$ also occurs for all $l = 1, \ldots, k$, then the number of components is exactly $\gcd(t_1, \ldots, t_k)$.

Theorem 2. *If there is a nonempty subset J of $K = \{1, \ldots, k\}$ such that*

$$\sum_{i \in K \setminus J} (n - t_i) < \gcd\{t_j \mid j \in J\} \tag{1}$$

then the Toeplitz graph $T_n\langle t_1, \ldots, t_k \rangle$ is not 2-connected, and if there is a nonempty subset J of $K = \{1, \ldots, k\}$ such that

$$\sum_{i \in K \setminus J} (n - t_i) < \gcd\{t_j \mid j \in J\} - 1 \tag{2}$$

then $T_n\langle t_1, \ldots, t_k \rangle$ is disconnected.

Proof. The Toeplitz graph H whose set of diagonals is $\{t_j \mid j \in J\}$ has at least $\gcd\{t_j \mid j \in J\}$ components. The graph $T_n\langle t_1, \ldots, t_k \rangle$ evolves from H by adding $n - t_i$ new edges for every $i \in K \setminus J$. Thus, if (2) holds, then $T_n\langle t_1, \ldots, t_k \rangle$ is disconnected, and if (1) holds, then $T_n\langle t_1, \ldots, t_k \rangle$ is not 2-connected. $\square$

For example, consider the graph $T_7\langle 3, 5, 6 \rangle$ (see Fig. 2) and let $J = \{1, 3\}$. Then Theorem 2 states that this graph is not 2-connected and hence nonhamiltonian. The next theorem describes a kind of degenerate case of Theorem 2 corresponding to $J = \emptyset$ with $\gcd \emptyset := n$.
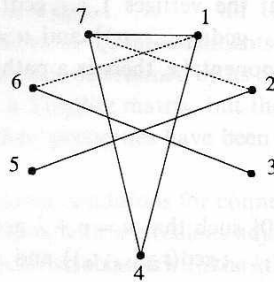
Theorem 3. *If*

$$\sum_{i=1}^{k} t_i > (k - 1)n$$

Fig. 2. The Toeplitz graph $T_7\langle 3, 5, 6 \rangle$.

then $T_n\langle t_1, \ldots, t_k \rangle$ *is not 2-connected and if*

$$\sum_{i=1}^{k} t_i > (k-1)n + 1$$

then $T_n\langle t_1, \ldots, t_k \rangle$ *is disconnected.*

Proof. The number of edges of $T_n\langle t_1, \ldots, t_k \rangle$ is

$$\sum_{i=1}^{k} (n - t_i) = kn - \sum_{i=1}^{k} t_i.$$

Therefore, under the first assumption the number of edges is less than n, and hence $T_n\langle t_1, \ldots, t_k \rangle$ is not 2-connected, and under the second assumption the number of edges is less than $n - 1$, and hence $T_n\langle t_1, \ldots, t_k \rangle$ is disconnected. $\square$

Theorem 4. *Consider the Toeplitz graph* $T_n\langle t_1, t_2 \rangle$ *and let* $n \geqslant 5$. *If*

$$t_1 + t_2 < n < 3t_1 + t_2 \tag{3}$$

then $T_n\langle t_1, t_2 \rangle$ *is nonhamiltonian. If* $t_1 \geqslant 3$ *and*

$$t_1 + t_2 + 2 < n < 3t_1 + t_2 \tag{4}$$

then $T_n\langle t_1, t_2 \rangle$ *is nontraceable.*

Proof. (3) is equivalent to the existence of two vertices u, v such that

$$1 \leqslant u \leqslant t_1, n - t_1 < v \leqslant n \quad \text{and} \quad v - u = t_1 + t_2.$$

Then both u and v have degree two, and any hamiltonian cycle of $T_n\langle t_1, t_2 \rangle$ contains the cycle $(u, u + t_1, v, u + t_2)$, which is impossible for $n \neq 4$.

If (4) holds, then there are three pairs of vertices u, v of the type described above, defining three different cycles. In each of these cycles, u and v, both having degree two, cannot be interior to a hamiltonian path of $T_n\langle t_1, t_2 \rangle$. Thus each of these three cycles contains an endpoint of the hamiltonian path, which is of course impossible. $\square$

2. Toeplitz graphs with $t_1 = 1$

Toeplitz graphs with $t_1 = 1$ are obviously traceable. Now we characterize those which are hamiltonian.

Lemma 1. *Let* $t_2 > n/2$. $T_n\langle 1, t_2, \ldots, t_k \rangle$ *is hamiltonian if and only if* $n, t_2, \ldots, t_k$ *are not all of the same parity.*
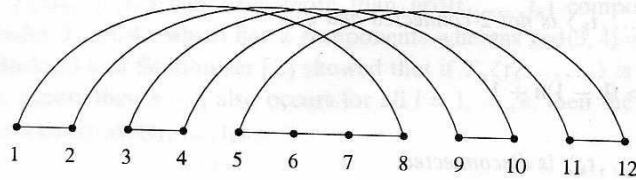
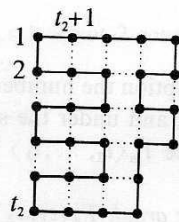Fig. 3. A hamiltonian cycle in the Toeplitz graph $T_{12}\langle 1, 7\rangle$.

Fig. 4. $T_n\langle 1, t_2\rangle$ with both n and t_2 even.

Proof. (if) If $n - t_i$ is odd for some i, $i = 2, \ldots, k$, then $(1, 1 + t_i, t_i, \ldots, n - t_i, n, n - 1,$ $n - t_i - 1, n - t_i - 2, n - 2, n - 3, n - t_i - 3, n - t_i - 4, \ldots, 2)$ is a hamiltonian cycle in $T_n\langle 1, t_2, \ldots, t_k\rangle$ (see Fig. 3 for the case $n = 12$, $t_2 = 7$).

(only if) Suppose now that $T_n\langle 1, t_2, \ldots, t_k\rangle$ is hamiltonian and $n - t_i$ is even for each i, $i = 2, \ldots, k$. Then any hamiltonian cycle of $T_n\langle 1, t_2, \ldots, t_k\rangle$ uses the path $P = n - t_2 + 1, n - t_2 + 2, \ldots, t_2 - 1, t_2$, because the vertices of P have degree two in $T_n\langle 1, t_2, \ldots, t_k\rangle$. Therefore, the graph H obtained from $T_n\langle 1, t_2, \ldots, t_k\rangle$ by contracting P to a single vertex is also hamiltonian. But H is also bipartite and has an odd number of vertices, namely $2(n - t_2) + 1$. This leads to a contradiction. □

Lemma 2. *Let $k = 2$ and $t_2 \leqslant n/2$. $T_n\langle 1, t_2\rangle$ is hamiltonian if and only if nt_2 is even.*

Proof. (only if) If $T_n\langle 1, t_2\rangle$ is hamiltonian, then both n and t_2 cannot be odd because in this case $T_n\langle 1, t_2\rangle$ is a bipartite graph with an odd number of vertices, hence nonhamiltonian.

(if) If nt_2 is even, then there are four cases to consider. In case both n and t_2 are even, a hamiltonian cycle of $T_n\langle 1, t_2\rangle$ is illustrated in Fig. 4. If n is odd and t_2 is even, then $T_n\langle 1, t_2\rangle$ is hamiltonian (see Fig. 5). If n is even and t_2 is odd, then either $\lfloor n/t_2\rfloor$ is even and Fig. 6 illustrates a hamiltonian cycle in $T_n\langle 1, t_2\rangle$, or $\lfloor n/t_2\rfloor$ is odd and such a cycle is given in Fig. 7. □

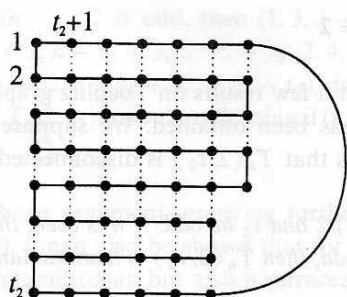By combining Lemmas 1 and 2 we get the desired characterization for the case $t_1 = 1$.

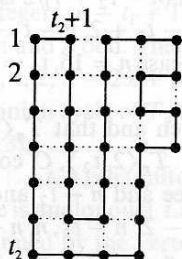Fig. 5. $T_n \langle 1, t_2 \rangle$ with n odd and t_2 even.

Fig. 6. $T_n \langle 1, t_2 \rangle$ with n even, t_2 odd and $\lfloor n/t_2 \rfloor$ even.
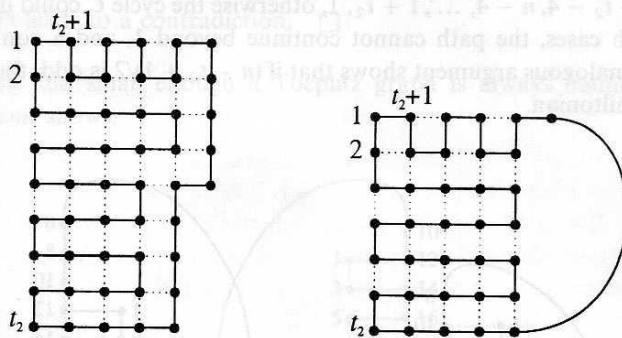

Fig. 7. $T_n \langle 1, t_2 \rangle$ with n even, t_2 odd and $\lfloor n/t_2 \rfloor$ odd.

Theorem 5. *The Toeplitz graph $T_n \langle 1, t_2, \ldots, t_k \rangle$ is hamiltonian if and only if either n, $t_2, \ldots, t_k$ are not all of the same parity, or they are all even and $t_2 \leqslant n/2$.*

Proof. Left to the reader. $\square$

3. Toeplitz graphs with $t_1 = 2$

In this section we present a few results on Toeplitz graphs with $t_1 = 2$. So far, no characterization theorem has been obtained. We suppose t_2 to be odd, otherwise $\gcd(2, t_2) > 1$, which implies that $T_n\langle 2, t_2 \rangle$ is disconnected.

Theorem 6. *Let $t_2 \geqslant (n - 1)/2$ and t_2 be odd. If n is even, then $T_n\langle 2, t_2 \rangle$ is traceable, but nonhamiltonian. If n is odd, then $T_n\langle 2, t_2 \rangle$ is hamiltonian if and only if $(n - t_2)/2$ is odd.*

Proof. Let n be even. If $(n - t_2 + 1)/2$ is even, then $1, 3, 5, \ldots, 2 + t_2, 2, 4, 4 + t_2, 6 + t_2, 6, 8, \ldots, n - 3, n - 1, n - t_2 - 1, n - t_2 + 1, n - t_2 + 3, \ldots, n - 2, n$ is a hamiltonian path in $T_n\langle 2, t_2 \rangle$, else, i.e. if $(n - t_2 + 1)/2$ is odd, $2, 4, 6, \ldots, 1 + t_2, 1, 3, 3 + t_2, 5 + t_2, 5, \ldots, n - 2, n, n - t_2, n - t_2 + 2, n - t_2 + 4, \ldots, n - 1$ is a hamiltonian path in $T_n\langle 2, t_2 \rangle$ (see Fig. 8 for the case $n = 16, t_2 = 9$ and $n = 16, t_2 = 11$, respectively).

Assume that $(n - t_2 + 1)/2$ is even and that $T_n\langle 2, t_2 \rangle$ has a hamiltonian cycle C. Since vertex n has degree two in $T_n\langle 2, t_2 \rangle$, C contains the path $n - t_2, n, n - 2$. Because $n - t_2 - 2$ has degree three and $n - t_2$ and $n - 2$ are two of its neighbors, C contains either the path $n - t_2 - 2, n - t_2, n, n - 2$, or the path $n - t_2, n, n - 2, n - t_2 - 2$. If C contains the path $n - t_2 - 2, n - t_2, n, n - 2$, then C also contains the path $1, 3, 5, \ldots, n - t_2 - 2, n - t_2, n, n - 2, n - 4, \ldots, 3 + t_2, 1 + t_2$, otherwise C would contain a smaller cycle. On the other hand, if C contains the path $n - t_2, n, n - 2, n - t_2 - 2$, then C also contains the path $n - t_2, n, n - 2, n - t_2 - 2, n - t_2 - 4, n - 4, \ldots, 1 + t_2, 1$, otherwise the cycle C could not be hamiltonian. In both cases, the path cannot continue beyond 1, and a contradiction is obtained. An analogous argument shows that if $(n - t_2 + 1)/2$ is odd, then $T_n\langle 2, t_2 \rangle$ is also nonhamiltonian.
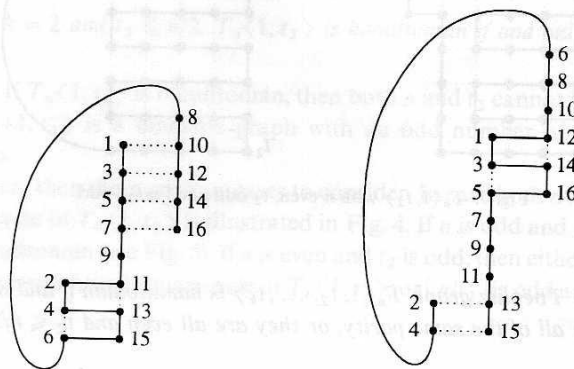

Fig. 8. The Toeplitz graphs $T_{16}\langle 2, 9 \rangle$ and $T_{16}\langle 2, 11 \rangle$.

Now let n be odd. If $(n - t_2)/2$ is odd, then $(1, 3, 3 + t_2, 5 + t_2, 5, 7, 7 + t_2, \ldots,$
$n - 1, n - t_2 - 1, n - t_2 + 1, n - t_2 + 3, \ldots, 2 + t_2, 2, 4, 4 + t_2, 6 + t_2, 6, 8, 8 + t_2,$
$\ldots, n - 2, n, n - t_2, n - t_2 + 2, n - t_2 + 4, \ldots, 1 + t_2)$ is a hamiltonian cycle in
$T_n \langle 2, t_2 \rangle$. The proof that $T_n(2, t_2)$ is nonhamiltonian if $(n - t_2)/2$ is even, is similar to
that in the case n is even. □

By a variant of the above argument, applying further ideas from the proof of
Theorem 9 (see Section 4), it can also be shown that for n odd and $(n - t_2)/2$ even,
$T_n \langle 2, t_2 \rangle$ is not only nonhamiltonian but also nontraceable.

Theorem 7. *Let n be even, $k > 2$, $t_2 \geqslant n/2$, t_i odd for all $i = 2, \ldots, k$. $T_n \langle 2, t_2, \ldots, t_k \rangle$ is
hamiltonian if and only if $(n - t_i + 1)/2$, $i = 2, \ldots, k$, are not all of the same parity.*

Proof. (if) Suppose two of the integers $(n - t_i + 1)/2$, $i = 2, \ldots, k$, say α and β, are of
different parity, for example α even and β odd. Then $(1, n - 2\beta + 2, n - 2\beta, \ldots, 2\alpha - 2,$
$n - 1, n - 3, 2\alpha - 4, 2\alpha - 6, \ldots, 4, 2, n - 2\alpha + 3, n - 2\alpha + 1, \ldots, 2\beta - 1, n, n - 2,$
$2\beta - 3, 2\beta - 5, \ldots, 3)$ is a hamiltonian cycle of $T_n \langle 2, t_2, \ldots, t_k \rangle$ (see Fig. 9 for the case
$n = 20$, $k = 3$, $t_2 = 11$ and $t_3 = 13$).

(only if) Suppose that $T_n \langle 2, t_2, \ldots, t_k \rangle$ is hamiltonian and all integers $(n - t_i + 1)/2$,
$i = 2, \ldots, k$, are even (the odd case is analogous). Let $\alpha = (n - t_2 + 1)/2$ and shrink the
subgraph of $T_n \langle 2, t_2, \ldots, t_k \rangle$ spanned by the vertex set $\{2\alpha + 1, 2\alpha + 3, \ldots, n - 1, 2,$
$4, \ldots, n - 2\alpha\}$ to a single vertex v. Obviously, the new graph H is also hamiltonian.
But H is bipartite, since its vertex set can be partitioned into the subsets $\{v, 1, 3 + t_2,$
$5, 7 + t_2, \ldots, n - t_2 - 2, n\}$ and $\{1 + t_2, 3, 5 + t_2, 7, \ldots, n - 2, n - t_2\}$ (see Fig. 10).
Moreover, H has an odd number of vertices, namely $2\alpha + 1$. Hence, H is nonhamiltonian, which leads to a contradiction. □

For t_2 odd and small enough a Toeplitz graph is always hamiltonian, as the
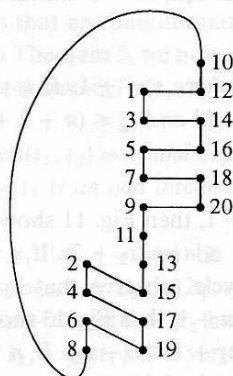following result shows.

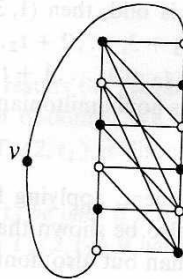Fig. 9. The Toeplitz graph $T_{20} \langle 2, 11, 13 \rangle$.
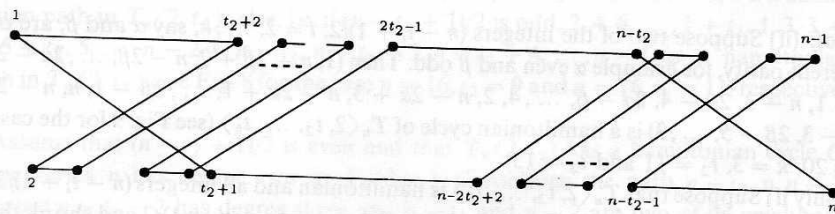
Fig. 10. Graph H.

Fig. 11. $T_n\langle 2, 4m+1\rangle$ with n even.

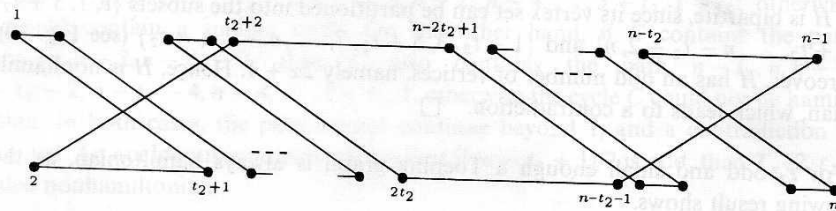Fig. 12. $T_n\langle 2, 4m-1\rangle$ with n even.

Theorem 8. *Let* $t_2 = \varepsilon \pmod 4$, *where* $\varepsilon = \pm 1$. *If* n *is even and* $t_2 \leqslant (n + \varepsilon)/3$, *then* $T_n\langle 2, t_2\rangle$ *is hamiltonian. If* n *is odd and* $t_2 \leqslant (n + 2 + \varepsilon)/4$, *then* $T_n\langle 2, t_2\rangle$ *is hamiltonian.*

Proof. If n is even and $t_2 = 4m + 1$, then Fig. 11 shows a hamiltonian cycle (observe that $2t_2 - 1 \leqslant n - t_2$ and $t_2 + 1 \leqslant n - 2t_2 + 2$). If n is even and $t_2 = 4m - 1$, then Fig. 12 shows a hamiltonian cycle (observe that $t_2 + 2 \leqslant n - 2t_2 + 1$ and $2t_2 \leqslant n - t_2 - 1$). If n is odd and $t_2 = 4m + 1$, then Fig. 13 shows a hamiltonian cycle (observe that $2t_2 - 1 \leqslant n - 2t_2 + 2$ and $t_2 + 1 \leqslant n - t_2$). If n is odd and $t_2 = 4m - 1$, then Fig. 14 shows a hamiltonian cycle (observe that $t_2 + 2 \leqslant n - t_2 - 1$ and $2t_2 \leqslant n - 2t_2 + 1$). $\square$
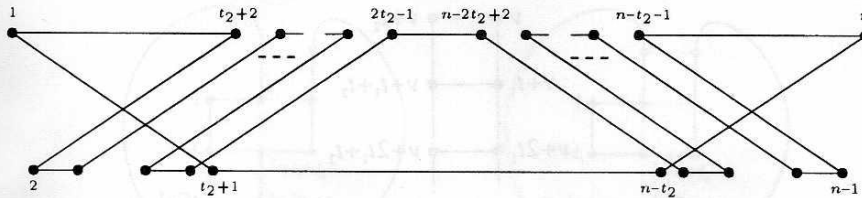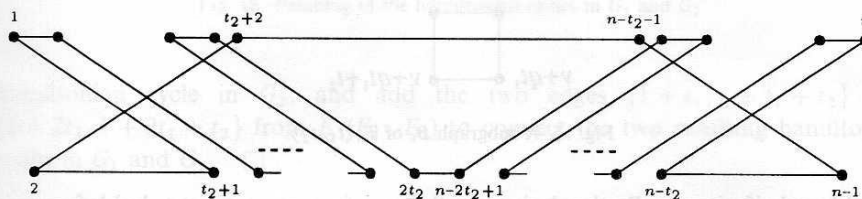
Fig. 13. $T_n\langle 2, 4m + 1\rangle$ with n odd.

Fig. 14. $T_n\langle 2, 4m - 1\rangle$ with n odd.

Corollary 2. *If t_2 is odd and less than $(n + 3)/4$, then $T_n\langle 2, t_2\rangle$ is hamiltonian.*

In Corollary 2, the only case not trivially covered by Theorem 8 is $(n + 1)/4 < t_2 < (n + 3)/4$, which implies $4t_2 = n + 2$. But in this situation n is even, $t_2 \geqslant 3$ and $n \geqslant 10$, from which $t_2 \leqslant (n - 1)/3$ follows and, by Theorem 8, $T_n\langle 2, t_2\rangle$ is hamiltonian. The example $n = 9$, $k = 2$, $t_1 = 2$, $t_2 = 3$ shows that the bound $(n + 3)/4$ mentioned in the corollary cannot be improved.

4. Other classes of Toeplitz graphs

In the case that $k = 2$ and t_1, t_2 satisfy the inequality $t_1 + 2t_2 \geqslant n$, we are able to characterize the Toeplitz graphs that are hamiltonian. Since the situation for $t_1 = 1$ has been completely described in Theorem 5, we now deal with $t_1 \geqslant 2$. In addition, we shall assume $\gcd(t_1, t_2) = 1$, for otherwise $T_n\langle t_1, t_2\rangle$ is disconnected, by Corollary 1.

Theorem 9. *Let $k = 2$, $t_1 \geqslant 2$, $\gcd(t_1, t_2) = 1$, and suppose $t_1 + 2t_2 \geqslant n$. $T_n\langle t_1, t_2\rangle$ is hamiltonian if and only if $(n - t_2)/t_1$ is an odd integer.*

Proof. (only if) For $v = 1, \ldots, t_1$ let $q = q(v)$ be the largest integer such that $v + qt_1 + t_2 \leqslant n$. Consider the subgraphs B_v on $2q + 2$ vertices as shown in Fig. 15. Notice that all vertices of B_v but $v + t_2$ and $v + qt_1$ have the same degree (2 or 3) both in B_v and $T_n\langle t_1, t_2\rangle$, thus $\{v + t_2, v + qt_1\}$ is a vertex cut-set.

Since we have $t_1 \geqslant 2$ such graphs B_v, $T_n\langle t_1, t_2\rangle$ is hamiltonian only if $v + t_2$ and $v + qt_1$ can be joined by a hamiltonian path in B_v, for all $v = 1, \ldots, t_1$, which is
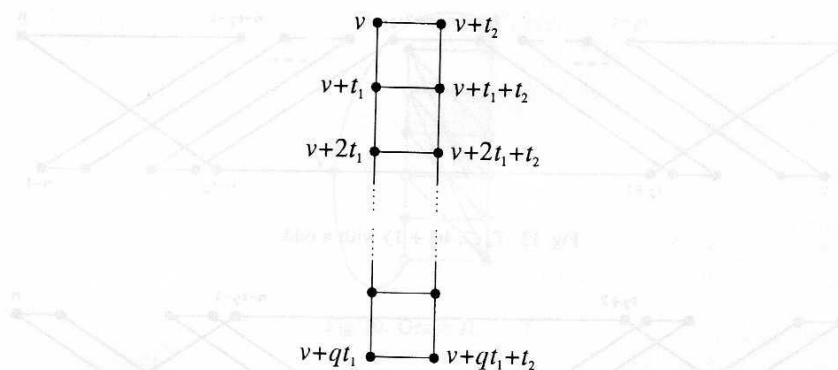
Fig. 15. A subgraph B_v of $T_n \langle t_1, t_2 \rangle$.

possible only if q is even. By the choice of q, $0 \leqslant n - (v + qt_1 + t_2) < t_1$ holds for every v, which means that $\lfloor (n - t_2 - v)/t_1 \rfloor$ is equal to q, and is thus even for any $v = 1, \ldots, t_1$. This occurs only if $(n - t_2)/t_1$ is an odd integer.

(if) On the other hand, if $(n - t_2)/t_1$ is an odd integer then q is even and therefore every subgraph B_v has a hamiltonian path P_v from $v + t_2$ to $v + qt_1$. Moreover, $T_n \langle t_1, t_2 \rangle$ contains t_1 further paths $Q_v = v + qt_1, v + (q + 1)t_1, \ldots, v + rt_1$, where r is the smallest integer such that $v + rt_1 > t_2$. Such a Q_v may be a single vertex ($r = q$) or a path of positive length, but in either case its endpoint $v + rt_1$ belongs to another subgraph $B_{v'}$ and plays there the role of $v' + t_2$. In this situation we have $v - v' = t_2 \pmod{t_1}$, i.e., Q_v joints B_v to B_{v+t_2}, and consequently the subgraphs are joined in the order $v, v + t_2, v + 2t_2, \ldots, v + (t_1 - 1)t_2 \pmod{t_1}$. Since $\gcd(t_1, t_2) = 1$, this yields a hamiltonian cycle in $T_n \langle t_1, t_2 \rangle$. $\square$

Another interesting situation occurs when n is a multiple of $t_1 + t_2$.

Theorem 10. *If* $\gcd(t_1, t_2) = 1$ *and* n *is a multiple of* $t_1 + t_2$, *then* $T_n \langle t_1, t_2 \rangle$ *is hamiltonian.*

Proof. We have to prove that the graph $T_n \langle t_1, t_2 \rangle = (V, E)$ with $|V| = n$ and $E = \{\{i, i + t_1\} \mid i = 1, 2, \ldots, n - t_1\} \cup \{\{i, i + t_2\} \mid i = 1, 2, \ldots, n - t_2\}$ is hamiltonian under the assumption that $n/(t_1 + t_2)$ is an integer k. We apply induction on k. By Theorem 9, the assertion is true for $k = 1$; suppose it has been verified for $k - 1$, $k \geqslant 2$. Consider two subgraphs of $T_n \langle t_1, t_2 \rangle$, namely $G_1 = (V_1, E_1)$ with $V_1 = \{1, 2, \ldots, t_1 + t_2\}$ and $E_1 = \{\{i, j\} \in E \mid i, j \in V_1\}$, and $G_2 = (V_2, E_2)$ with $V_2 = \{t_1 + t_2 + 1, \ldots, n\}$ and $E_2 = \{\{i, j\} \in E \mid i, j \in V\}$. By Theorem 9 and by the induction hypothesis, both G_1 and G_2 are hamiltonian. Moreover, G_1 is itself a cycle and $t_1 + t_2 + 1$ has degree 2 in G_2.

To obtain a hamiltonian cycle in $T_n \langle t_1, t_2 \rangle$ we remove two edges, namely $\{1 + t_1, 1 + 2t_1\}$ from the cycle G_1 and $\{1 + t_1 + t_2, 1 + 2t_1 + t_2\}$ from the
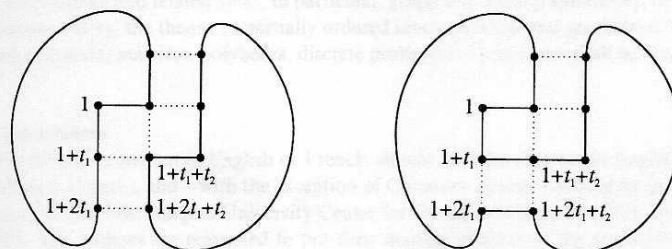
Fig. 16. Patching of the hamiltonian cycles in G_1 and G_2.

hamiltonian cycle in G_2, and add the two edges $\{1 + t_1, 1 + t_1 + t_2\}$ and $\{1 + 2t_1, 1 + 2t_1 + t_2\}$ from $E\backslash(E_1 \cup E_2)$ to connect the two resulting hamiltonian paths in G_1 and G_2. □

Acknowledgements

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