Intersections of Longest Cycles in Grid Graphs

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Received August 5, 1996

Abstract: It is well-known that the largest cycles of a graph may have empty intersection. This is the case, for example, for any hypohamiltonian graph. In the literature, several important classes of graphs have been shown to contain examples with the above property. This paper investigates a (nontrivial) class of graphs which, on the contrary, admits no such example. J Graph Theory **25:** 37–52, 1997

1. INTRODUCTION

In 1966 T. Gallai (see [1]) asked whether there exist connected graphs in which the intersection of all longest paths is empty. The first example, provided by H. Walther [8] in 1969, was rather involved, but showed that the case of paths does not qualitatively differ from that of cycles, for which Petersen's graph was a notorious example. Subsequently the question was refined in [9] and answered to some extent in a series of papers [2], [3], [5], [10], [11], [12] (see Chapter 3.12 in H.-J. Voss' book [7]). By restricting the class of graphs under consideration we can guarantee that the intersection of all longest paths or cycles necessarily be nonempty. This is the case, for example, for planar 4-connected graphs because, by a famous theorem of Tutte [6], they are all hamiltonian.

The lattice graphs—finite connected subgraphs of the usual infinite square lattice graph in $\mathbf{R}^2$—form without any doubt an important family from both the pure and applied mathematical

*Author was partly supported by a grant from the City University of New York PSC/CUNY Research Award Program.

 CCC 0364-9024/97/010037-16

points of view. In particular, the hamiltonian problem in lattice graphs is as nontrivial as in the general case. In [4], B. Menke showed that, among the lattice graphs too, we can find examples whose longest cycles have empty intersection. However, for a very natural restriction of this class of graphs, he showed that no such example exists. The graphs of the restricted class can be defined as follows: We take a cycle in the infinite square lattice graph and then consider all points and edges lying inside or on that cycle. We call a graph obtained in this way a *grid graph.*

B. Menke [4] proved that every grid graph possesses at least 4 points which lie on every longest cycle. Our aim here is to complement this qualitative result by some quantitative ones depending only on two special point sets on the cycle defining the grid graph. We shall see as a consequence that in many grid graphs the longest cycles have a rather large intersection.

Also, the proof in [4] had a global character. We shall provide here a useful local theorem stating that the intersection of all longest cycles contains points close to any pair of adjacent points, one of degree 2 and the other of degree 3.

Let G be a grid graph, defined by a cycle Γ which will be called the *boundary cycle* of G. Let A be the set of all points of G of degree 2 and B the set of all points of Γ of degree 4 in G. Let $\rho(A)$ denote the smallest distance between any two different points in A, and $\rho(A, B)$ the smallest distance between any point in A and any point in B. If $B = \emptyset$, put $\rho(A, B) = \infty$. Also, let Z be the point set of the intersection of all longest cycles of G.

Moreover, we denote by S the (unique) grid graph with $|V(S)| = 9$.

2. FOUR LEMMAS

We give here four lemmas. The third is the basis for our local Theorem 1, while the last is used in the proof of Theorem 4.

Lemma 1 [4]. Any path in G of length 2 whose points are collinear meets all longest cycles of G.

Lemma 2. If Γ contains the path [2, 1, 3], where $d_G(1) = 2, d_G(2) \geq 3, d_G(3) \geq 3$ and 4 is the other common neighbor of 2 and 3 then each longest cycle of G meets the set $\{1, 2, 3, 4\}$.

Proof. Using the notation of Figure 1, $d_G(1) = 2, d_G(2) \geq 3$ and $d_G(3) \geq 3$ yield 5, 7, 8 $\in V(G)$. The other points of the lattice shown on Figure 1, which may or may not belong to $V(G)$, are introduced for immediate or later use. Each of the Figures 2–15 and 17–24 contains two configurations. The indicated sets of vertices are the same in both configurations.

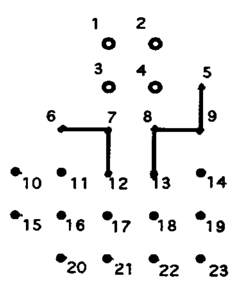

Figure 1

Suppose the points 1, 2, 3, 4 are missed by some longest cycle C. Then, by Lemma 1, we have 5, 7, $8 \in V(C)$.

Let P denote [7, 8] or [7, 12, 13, 8]. If $P \subset C$ then

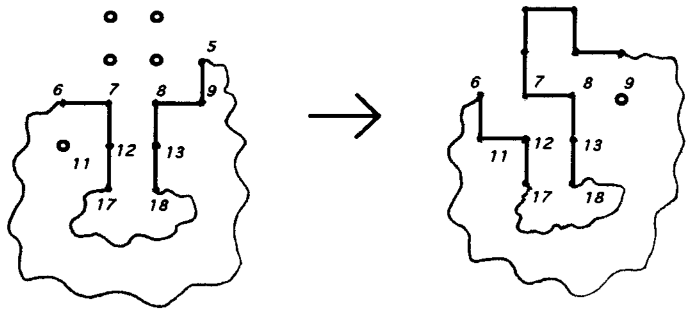

Figure 2a

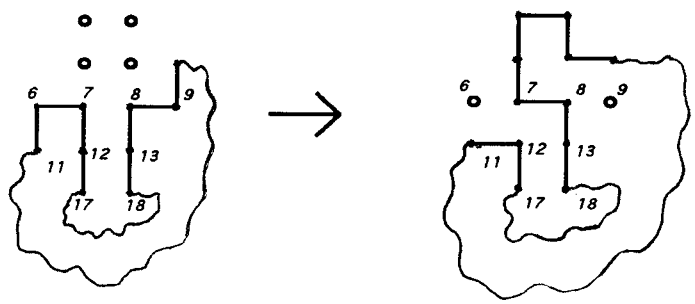

Figure 2b

P could be replaced by [7, 3, 1, 2, 4, 8], which would yield a longer cycle in G. So $P \not\subset C$ and

$$[6, 7, 12] \cup [13, 8, 9] \subset C.$$

In particular, $(8, 9) \in E(C)$; hence, by symmetry, $(5, 9) \in E(C)$ too. If $(11, 12) \in E(C)$, then replacing [6, 7, 12, 11] by [6, 11] and [13, 8, 9, 5] by [13, 12, 7, 3, 4, 8, 9, 5] yields again a longer cycle. Hence $(11, 12) \notin E(C)$; analogously $(13, 14) \notin E(C)$. Since $(12, 13) \notin E(C)$ too,

$$[6, 7, 12, 17] \cup [18, 13, 8, 9, 5] \subset C.$$

Figures 2a and 2b show how to obtain a longer cycle if $11 \notin V(C)$ or $(6, 11) \in E(C)$. Therefore $[10, 11, 16] \subset C$.

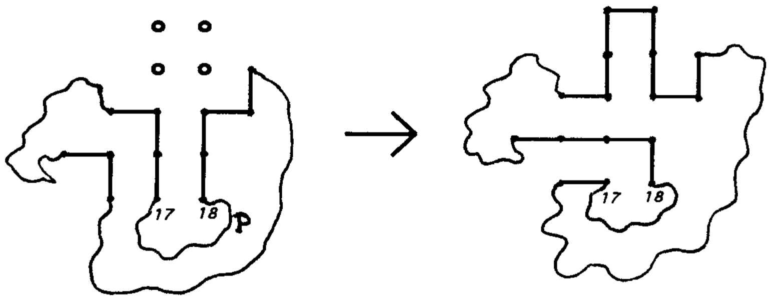

Figure 3

Figure 3 shows why $P \not\subset C$, where P is any path joining 17 with 18 and avoiding the points 6, 7, 12, 13, 8, 9, 5, 10, 11, 16.

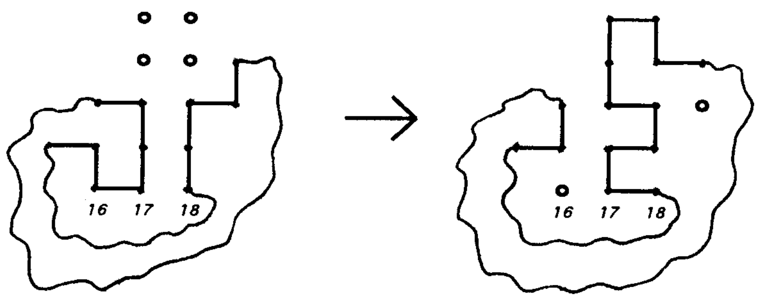

Figure 4

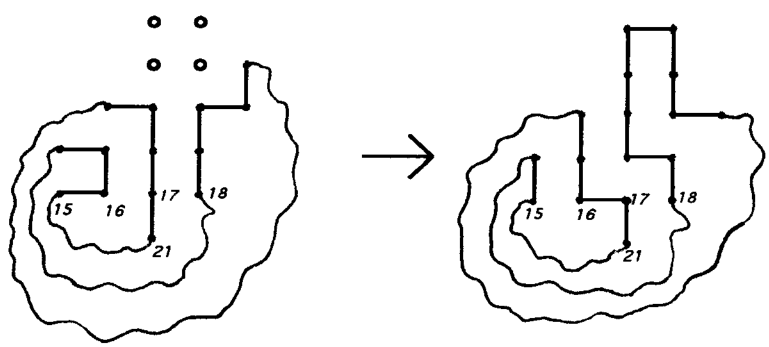

Figure 5

Figure 4 reveals that $(16, 17) \notin E(C)$. It follows that $(17, 21) \in E(C)$.

We see on Figure 5 why $(15, 16) \notin E(C)$, whence $(16, 20) \in E(C)$.

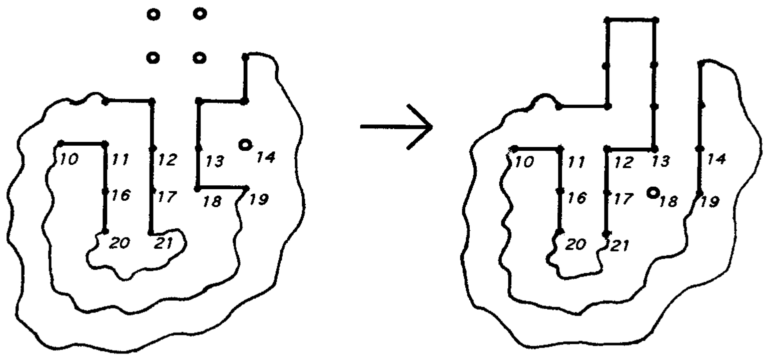

Figure 6a

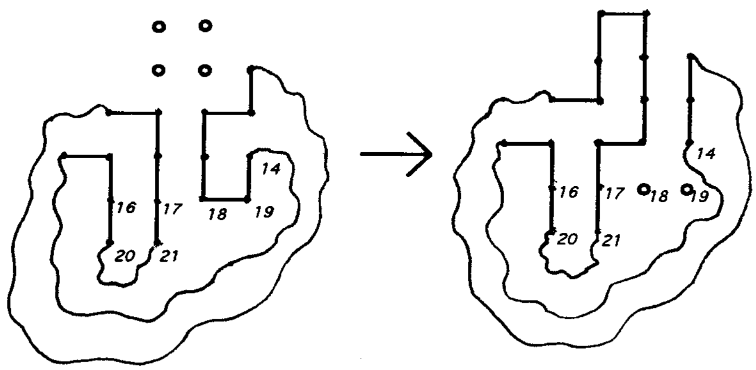

Figure 6b

Supposing $(18, 19) \in E(C)$, Figures 6a and 6b show cycles longer than C in both cases $14 \notin V(C)$ and $14 \in V(C)$.

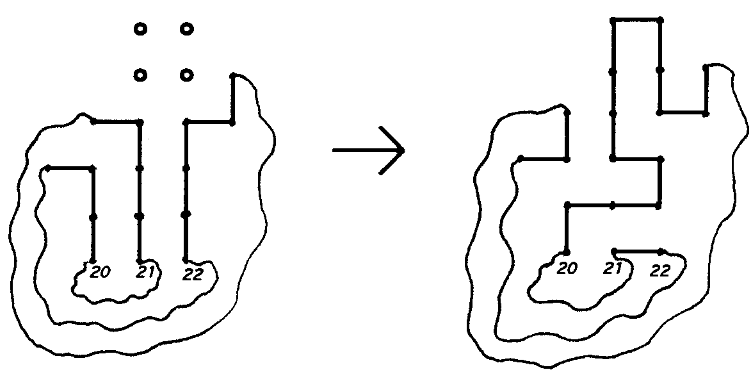

Figure 7

Therefore $(18, 22) \in E(C)$, and in this case Figure 7 shows a longer cycle. This contradiction proves Lemma 2. ∎

Lemma 3. If Γ contains the path [2, 1, 3], $d_G(1) = 2, d_G(2) \geq 3$ and $d_G(3) \geq 3$ then each longest cycle of G meets the set $\{1, 2, 3\}$.

Proof. Suppose the points 1, 2, 3 are missed by some longest cycle C of G, see Figure 8.

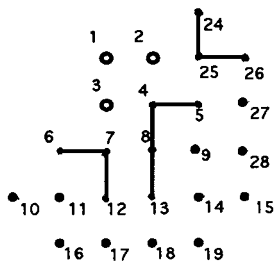

Figure 8

By Lemmas 1 and 2, the points 7, 4, and 25 belong to C and obviously $(7, 8), (5, 25) \notin E(C)$. Thus

$$[6, 7, 12] \cup [8, 4, 5] \cup [24, 25, 26] \subset C.$$

Either (8, 13) or (5, 27) belongs to $E(G)$ (if not, the square with vertices 4, 5, 9, 8 would lie in C, which is impossible), say $(8, 13) \in E(G)$.

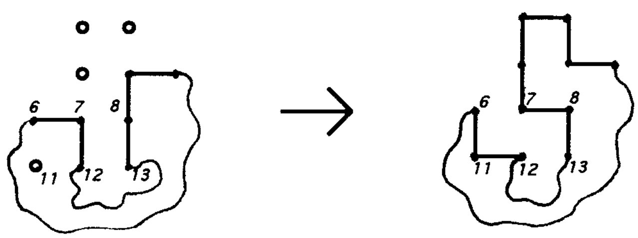

Figure 9a

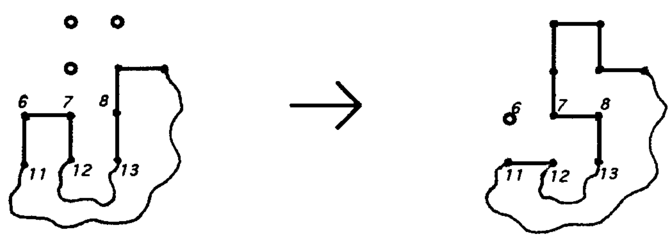

Figure 9b

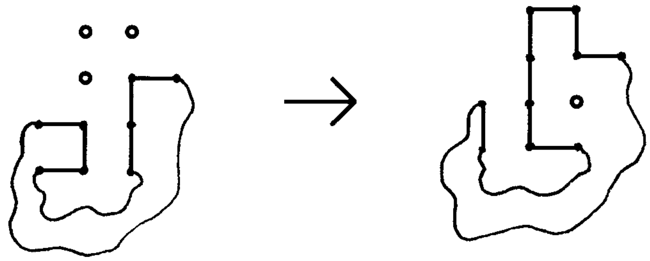

Figure 9c

Figure 9a shows that $11 \in V(C)$, while Figures 9b and 9c show that $(6, 11) \notin E(C)$ and $(11, 12) \notin E(C)$, respectively. Therefore $[10, 11, 16] \subset C$.

Concerning the edge (12, 13) there are two cases to consider.

Case I. $(12, 13) \in E(C)$.

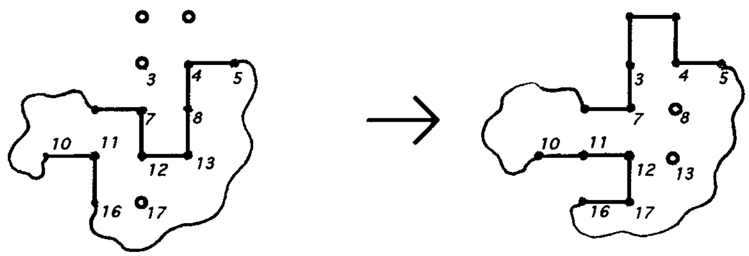

Figure 10a

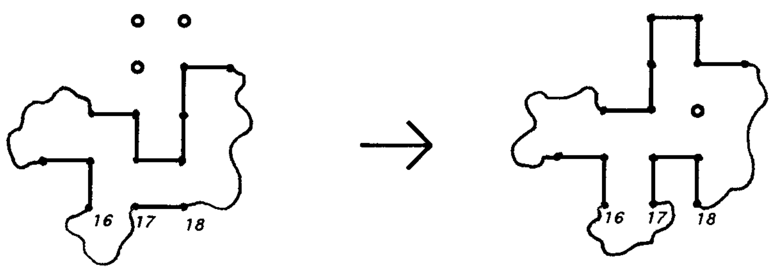

Figure 10b

If $17 \notin V(C)$ then Figure 10a shows a cycle longer than C. Figure 10b shows why $(17, 18) \notin E(C)$. Thus $[16, 17, 21] \subset C$.

The point 18 cannot be missed by C, as Figure 11 illustrates. This together with $(17, 18) \notin E(C)$ yields $[22, 18, 19] \subset C$.

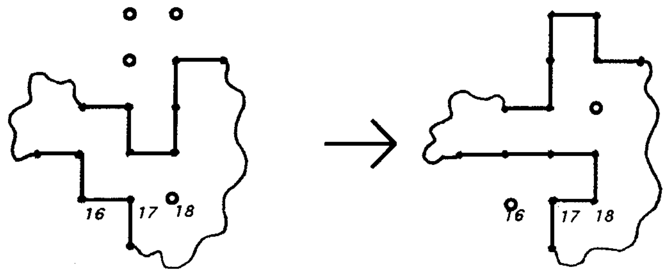

Figure 11

If $14 \notin V(C)$, a longer cycle is shown in Figure 12a. Also $(9, 14) \notin E(C)$, otherwise see Figure 12b. Hence $[19, 14, 15] \subset C$.

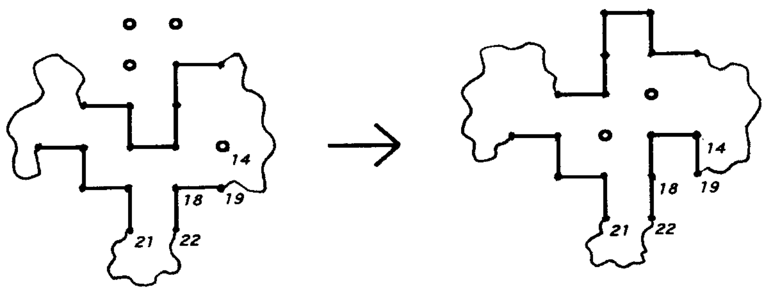

Figure 12a

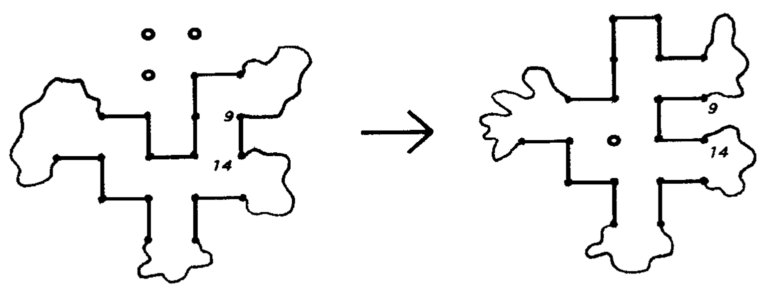

Figure 12b

Supposing now $9 \notin V(C)$, we discover a longer cycle in G, as Figure 13 reveals. Therefore $[5, 9, 28] \subset C$.

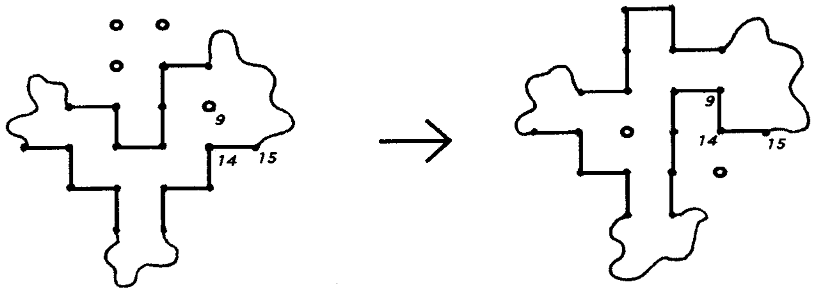

Figure 13

Figures 14a and 14b show that $27 \in V(C)$ and $(27, 28) \notin E(C)$.

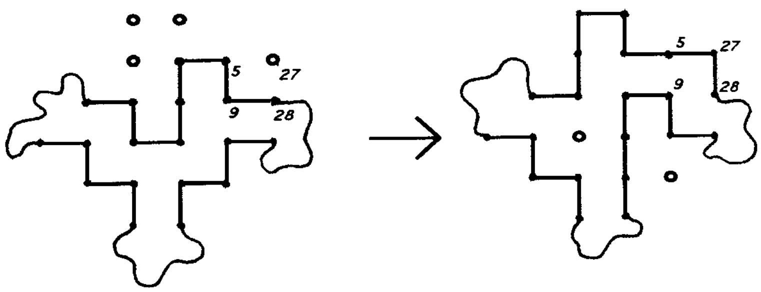

Figure 14a

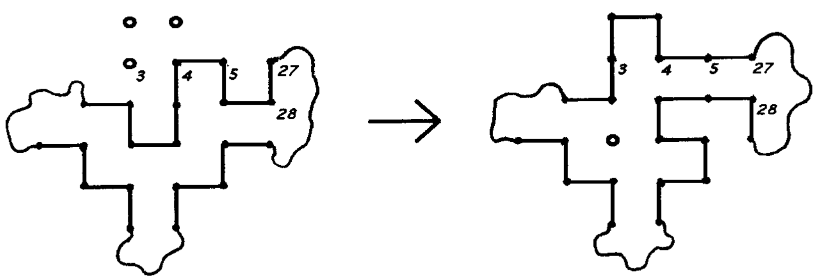

Figure 14b

It follows that $[26, 27, 29] \subset C$, but in this case Figure 15 provides a longer cycle. Here the cycle C is not entirely shown because there exist several combinatorial possibilities for C to include the paths appearing in the figures.

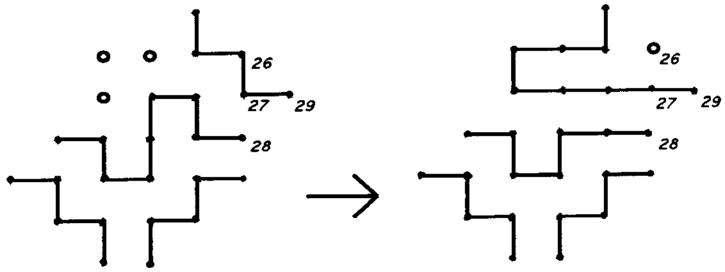

Figure 15

Case II. $(12, 13) \notin E(C)$.

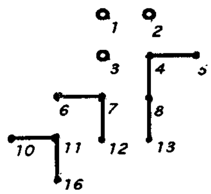

Figure 16

We start again with the situation shown in Figure 16. We must have $(12, 17) \in E(C)$.

Now, if $(13, 14) \in E(C)$, Figures 17a, 17b and 17c show cycles longer than C, according to whether $9 \notin V(C)$, $(9, 14) \in E(C)$ or $(5, 9) \in E(C)$. It follows that $(13, 18) \in E(C)$.

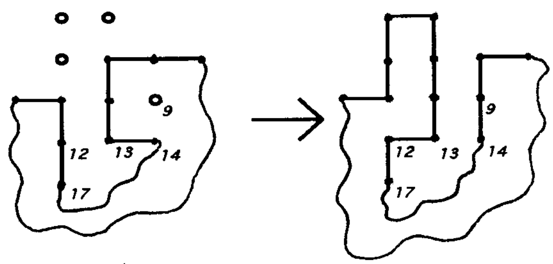

Figure 17a

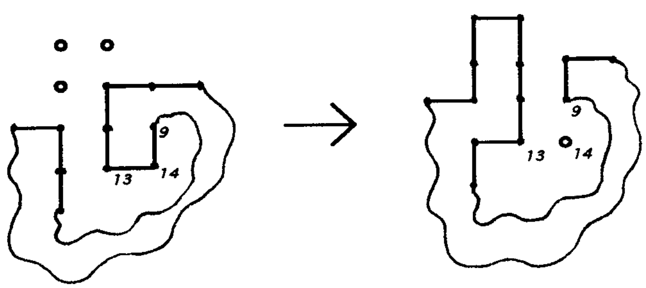

Figure 17b

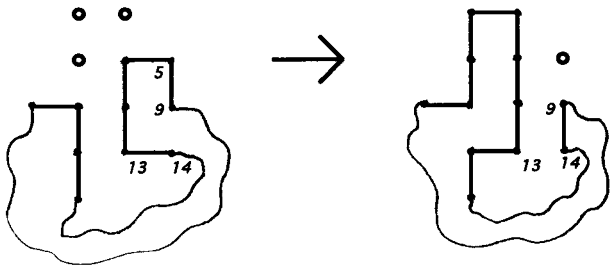

Figure 17c

We see now that there is no path $P \subset C$ joining 17 with 18 and avoiding 10, 11, 16, 6, 7, 12, 13, 8, 4, 5. Indeed if such a path existed Figure 18 would show a cycle longer than C.

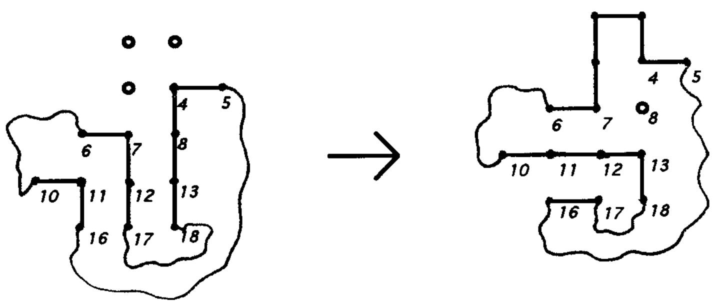

Figure 18

Also, $(16, 17) \notin E(C)$, as follows from inspecting Figure 19. Thus, $(17, 21) \in E(C)$.

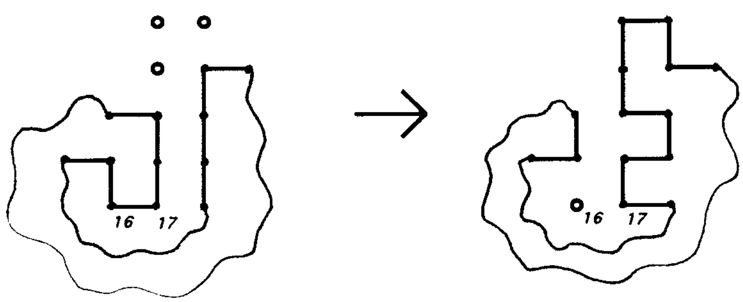

Figure 19

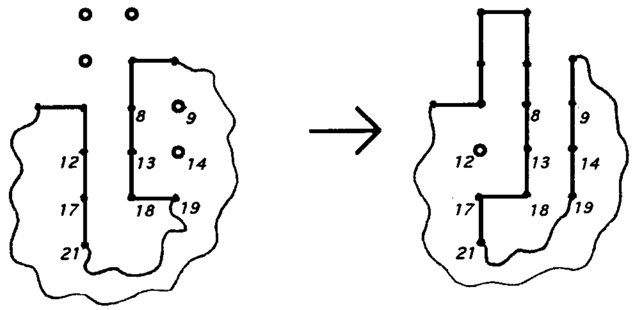

Figure 20

Suppose now $(18, 19) \in E(C)$. If $9, 14 \notin V(C)$, Figure 20 shows a longer

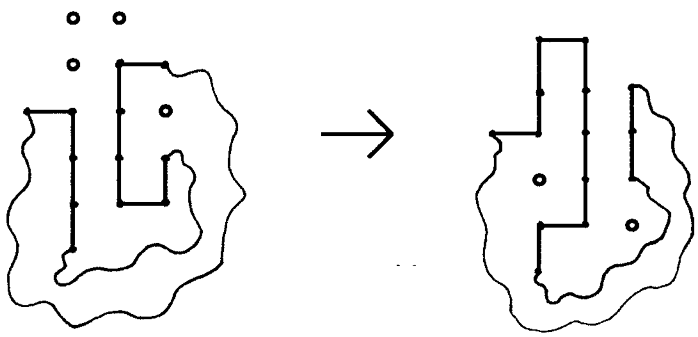

Figure 21a

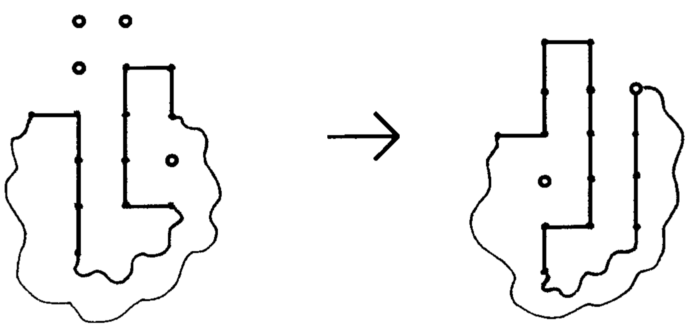

Figure 21b

cycle. If only one of these points is not on C, Figure 21 shows longer cycles.

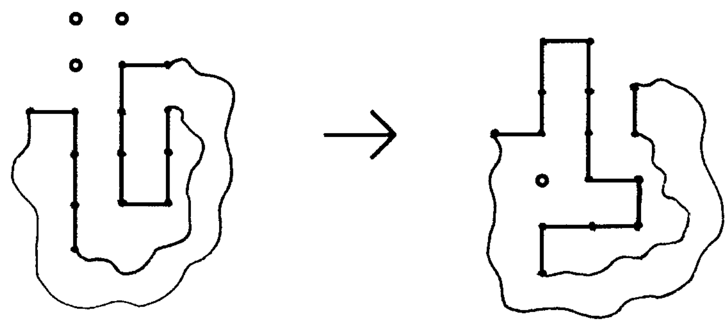

Figure 22a

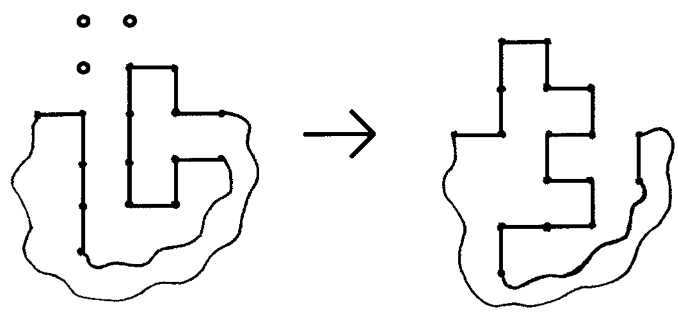

Figure 22b

If both belong to C, there are four different situations, for which Figure 22 provides

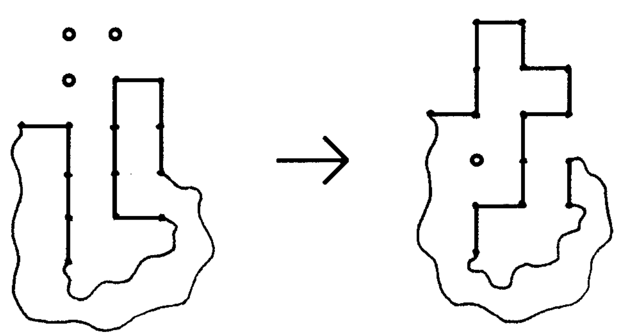

Figure 22c

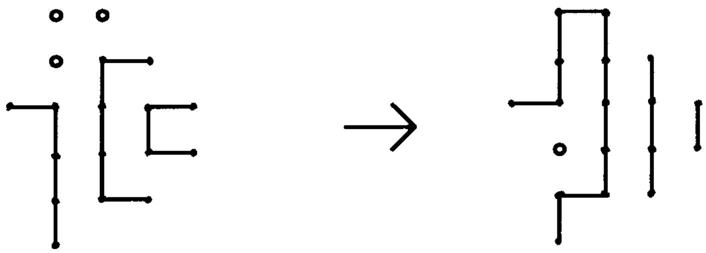

Figure 22d

cycles longer than C. Hence $(18, 22) \in E(C)$. But then Figure 23 indicates

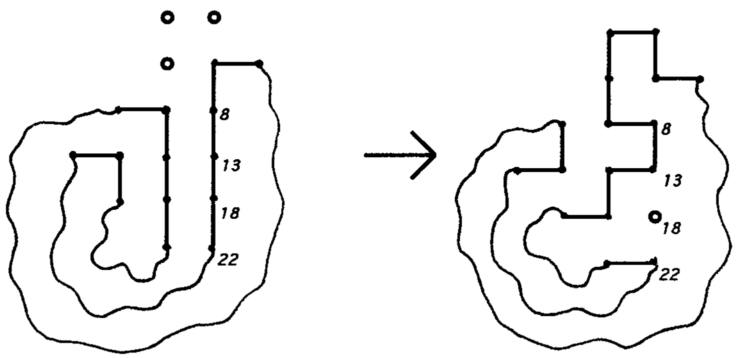

Figure 23

a longer cycle! With this final contradiction the proof ends. ∎

The different situation Lemma 4 deals with makes a new notation necessary. This will be incompatible with the above (note that Lemmas 2 and 3 had compatible notations).

Lemma 4. If Γ contains the path [5, 3, 1, 2, 4, 6], $d_G(1) = d_G(2) = 2$ and $d_G(3) = d_G(4) = 3$ then each longest cycle of G passes through 5 and 6.

Proof. If, for some longest cycle C of $G, 5 \notin V(C)$ then $1, 2, 3, 4 \notin V(C)$ either. This contradicts Lemma 1. Analogously, $6 \notin V(C)$ is impossible. ∎

3. RESULTS

Again, the notation will be different from those used in the preceding section.

Theorem 1. If Γ contains the path [2, 1, 3], $d_G(1) = 2, d_G(2) = 3$ and $d_G(3) \geq 3$ then each longest cycle of G passes through 3.

Proof. Suppose there is a longest cycle C missing 3. Then $1 \notin V(C)$ and, by Lemma 3, $2 \in V(C)$, which implies $[4, 2, 5] \subset C$, see Figure 24.

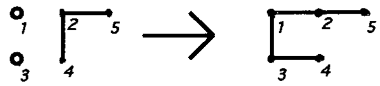

Figure 24

But then replacing [4, 2] by [4, 3, 1, 2] yields a cycle longer than C, which is a contradiction. ∎

Theorem 2. If $\rho(A,B) \geq 2$ and $\rho(A) \geq 3$ then $|Z| \geq 2|A|$.

Proof. For every point x of G of degree 2 consider the two points α_x, β_x adjacent to x(x and α_x have equal ordinates). Since $\rho(A) \geq 3$, it is not possible to find different points $x, y \in A$ such that $\alpha_x = \beta_y$ or $\alpha_y = \beta_x$ or $\alpha_x = \alpha_y$ or $\beta_x = \beta_y$. This implies that precisely $2|A|$ points of G are adjacent to points in A. Because $\rho(A,B) \geq 2$, all these points have degree 3. By Theorem 1, they lie on each longest cycle of G. ∎

Theorem 3. If $\rho(A,B) \geq 2$ and $\rho(A) \geq 2$ then $|Z| \geq \frac{3}{2}|A|$ or G is isomorphic to S.

Proof. Let $a_1, a_2, \ldots, a_k$ be the points of Γ of degree 2 in G, in cyclic order ($a_{k+1} = a_1$). Clearly $k \geq 4$. Unless G is isomorphic to S, no three points a_i, a_{i+1}, a_{i+2} can satisfy $d(a_i, a_{i+1}) = d(a_{i+1}, a_{i+2}) = 2, d$ denoting here the distance between points in G. Therefore there are $p \leq k/2$ pairwise disjoint pairs of consecutive points of A such that the distance in Γ between any two points in A which do not belong to the same pair is at least 3. For every such pair the number of neighbors in Γ is 3, while for any other point of A there are 2 neighbors. Thus there are precisely

$$3p + 2(k - 2p) = 2k - p$$

points in Γ adjacent to points in A. By Theorem 1, all these points belong to Z, whence

$$|Z| \geq 2k - p \geq 2k - \frac{k}{2} = \frac{3}{2}k.$$

∎

Theorem 4. If $\rho(A,B) \geq 2$ then $|Z| \geq |A|$.

Proof. For $G = C_4$ the statement is true. Suppose now that $G \neq C_4$ and let again $A = \{a_1, a_2, \ldots, a_k\}$. Clearly, for no index $i, d(a_i, a_{i+1}) = d(a_{i+1}, a_{i+2}) = 1$. From A we select p pairs of points adjacent in Γ such that no two other points of A be adjacent in Γ. We shall define a function $f\colon A \to V(\Gamma)$.

For any point $x \in A$ not belonging to a selected pair, the point γ_x of Γ adjacent to x and preceding it in a chosen cyclic order on Γ is, by Theorem 1, in Z. Put $f(x) = \gamma_x$.

For any adjacent points $a_i, a_{i+1} \in A$, the points $c_i, c_{i+1} \in \Gamma$ with

$$d(c_i, a_i) = d(c_{i+1}, a_{i+1}) = 2$$

and

$$d(c_i, a_{i+1}) = d(c_{i+1}, a_i) = 3$$

lie, by Lemma 4, in Z.

Suppose now for two consecutive selected pairs a_i, a_{i+1} and a_{i+2}, a_{i+3}, we have $c_{i+1} = c_{i+2}$, where c_{i+j} is the point at distance j from $a_{i+j} (j = 2, 3)$ on Γ. By Lemma 4, $c_i, c_{i+1}, c_{i+3} \in Z$, where c_{i+3} is the point of Γ at distance 2 from a_{i+3} and 3 from a_{i+2}. Let b_{i+2} be the point adjacent to both a_{i+j} and $c_{i+j} (j = 0, 1, 2, 3)$. We show that $b_i, b_{i+1}, b_{i+2}, b_{i+3} \in Z$. By symmetry it is enough to prove this for b_i, b_{i+1}. Suppose indeed $b_{i+1} \notin Z$. Then a longest cycle C of G has c_i, c_{i+1} or $c_{i+3}, c_{i+1}, b_{i+2}$ as consecutive points. In the first case C would be extendable to a longer cycle by replacing the edge (c_i, c_{i+1}) by the path $[c_i, b_i, b_{i+1}, c_{i+1}]$. In the second case the length of C would be 6, while the circumference of G is obviously at least 12. Hence $b_{i+1} \in Z$, which also implies $b_i \in Z$. Put

$$f(a_{i+j}) = \begin{cases} b_{i+j} \text{ if } c_{i+1} = c_{i+2} \\ c_{i+j} \text{ if } c_{i+1} \neq c_{i+2} \end{cases}$$

$(j = 0, 1, 2, 3)$. Since $\rho(A, B) \geq 2$, for every selected pair a_i, a_{i+1} and every point $x \in A$ not belonging to any selected pair, b_i, b_{i+1}, c_i and c_{i+1} are distinct from γ_x. Hence f is injective and $|Z| \geq k$. ∎

ACKNOWLEDGMENTS

Thanks are due the referees, who contributed to a better final version of this paper.

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