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Closed geodesic arcs in Aleksandrov spaces

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Introduction

In this paper a metric space $(\mathcal{A}, \delta)$ is called an Aleksandrov space if it is a complete intrinsic metric space with curvature bounded below by some real number k in the sense of A. D. Aleksandrov.

In Riemannian manifolds there is a certain connection between the conjugate points, the closed geodesic arcs, and the cut locus. This connection is described, for instance, by the following theorem from S. Kobayashi's paper [4].

Theorem K. *Let M be a Riemannian manifold, $x \in M$ and y be a point of the cut locus of x closest to x. Then either y is conjugate to x, or y is the midpoint of a geodesic arc starting and ending at x.*

We are going to strengthen this theorem in several directions and generalize it to arbitrary Aleksandrov spaces.

Definitions and notation

Let S_k denote the 2-dimensional complete simply-connected Riemannian manifold of curvature $k \in \mathbb{R}$.

For $k \in \mathbb{R}$, a complete intrinsic metric space $(\mathcal{A}, \delta)$ such that every point of $\mathcal{A}$ has a neighbourhood in which any four points admit an isometric embedding in $S_{k'}$ for some $k' \geq k$ is called a space with curvature bounded below by k in the sense of A. D. Aleksandrov, for short an *Aleksandrov space.*

If $a, b, c \in \mathcal{A}$, let $\angle^* abc$ denote the angle of the triangle in S_k of side-lengths $\delta(a, b)$, $\delta(b, c)$, $\delta(c, a)$, opposite to the side of length $\delta(c, a)$.

Berestovskii [1] proved that the intrinsic metric space $(\mathcal{A}, \delta)$ is an Aleksandrov space if and only if every point of $\mathcal{A}$ has a neighbourhood in which, for any four distinct points a, b, c, d, we have

$$\angle^* bac + \angle^* cad + \angle^* dab \leq 2\pi.$$

For $k > 0$, the definition requires in addition that, if $\mathcal{A}$ is a one-dimensional manifold, then diam $\mathcal{A} \leq \pi/\sqrt{k}$.

For several other characterizations of Aleksandrov spaces, consult Burago, Gromov and Perelman's work [2].

A *segment* in $\mathcal{A}$ is a shortest path between two points of $\mathcal{A}$.

A *geodesic* in $\mathcal{A}$ is the image of an interval $I \subset \mathbb{R}$ through a continuous mapping $c : I \to \mathcal{A}$, such that every point in I has a neighbourhood in I on which c is an isometry. If $I = \mathbb{R}$ and c is periodic then $c(I)$ is called a *closed geodesic.* If I is a compact interval $[a, b]$ then $c(I)$ is called a *geodesic arc*; if, moreover, $c(a) = c(b)$ then $c(I)$ is said to be a *closed geodesic arc at* $c(a)$.

We use some basic concepts and results developed in [2]. So, for example, in any Aleksandrov space geodesics have definite directions and do not branch. Moreover, the angle between two geodesics exists.

If pa, pb are geodesics, let $\angle apb$ denote the angle between pa and pb.

Now consider an arbitrary (but nonempty) maximal (by inclusion) geodesic $G = c(I)$ containing $x = c(s)$. Then G has at least one direction $+$ at x, the one corresponding, say, to the direction of increasing numbers in I. Let $t > s$ be a point of I such that, for some neighbourhood $\mathcal{N}$ of the subgeodesic $G_y = c([s, t])$ of G from x to $y = c(t)$, the minimal length of arcs from x to y in $\mathcal{N}$ is realized uniquely by G_y. The existence of such a point t is guaranteed by the definition of a geodesic in conjunction with the basic property that a proper subarc of a segment realizes uniquely the minimum distance between its endpoints. Denote by Ψ the set of all these points t and let $x_+ = c(\inf\{t > s : t \notin \Psi\})$, allowing for x_+ the value $+\infty$ too, in case $t \in \Psi$ for all $t > s$ (which will never occur if G is a geodesic arc). We call x_+ the *conjugate* of x on G in direction $+$. This corresponds to the differential geometric notion of first conjugate point.

An *endpoint* in $\mathcal{A}$ is a point not interior to any segment (or, equivalently, to any geodesic).

For $x \in \mathcal{A}$, the *cut locus* $C(x) \subset \mathcal{A}$ is the set of all points $y \in \mathcal{A} \setminus \{x\}$ such that there is a segment from x to y, not extendable as a segment beyond y.

Auxiliary results

We start with a few lemmas. The first two can be found in Burago, Gromov and Perelmen's paper [2].

Lemma 1 [2]. *If the point p is interior to the geodesic ab and $q \notin ab$, then $\angle qpa + \angle qpb = \pi$.*

An important result obtained in [2] is the following generalized Toponogov theorem.

Lemma 2 [2]. *For any geodesic triangle abc in an Aleksandrov space,*

$$\angle^* abc \leq \angle abc, \quad \angle^* bca \leq \angle bca, \quad \angle^* cab \leq \angle cab.$$

If $p \in \mathcal{A}$, then the *space* Σ_p *of directions* at the point p is defined as the completion of the metric space consisting of classes of geodesics starting at p, all geodesics in a class overlapping, and the distance being the angle between representatives (see [2], p. 23).

An important example of an Aleksandrov space is a convex surface. The space $\mathcal{S}$ of all (nondegenerate) closed convex surfaces in $\mathbb{R}^3$, endowed with the well-known Pompeiu-Hausdorff metric is a Baire space.
"Most" means "all, except those in a first Baire category set".

Lemma 3 [5]. *On most closed convex surfaces, most points are endpoints.*

For other geometrically relevant, curious phenomena on convex surfaces, obtained via Baire categories, see [3], [6], [8].

Main result

To prove our main theorem we also need the following lemma, well-known in the Riemannian case, and also proven for convex surfaces in [7]. For the reader's convenience, we formulate and prove it here in the frame of Aleksandrov spaces.

Lemma 4. *If in an Aleksandrov space $\mathcal{A}$ a point $y \in C(x)$ can be joined with x by a single segment only, then y is conjugate to x.*

Proof. Consider an open neighbourhood $\mathcal{N}$ of Γ (in the space of arcs) and an extension Γ_1 of Γ beyond y in $\mathcal{N}$. Since $y \in C(x)$, any segment Γ_1'

joining x with the endpoint y_1 of Γ_1 different from x is different from Γ_1. Take $y_n \in \Gamma_1 \setminus \Gamma$ $(n = 2, 3, ...)$ such that $y_n \to y$, and consider the subarc Γ_n of Γ_1 joining x to y_n, and the segment Γ'_n from x to y_n, which must be different from Γ_n. No subsequence of $\{\Gamma'_n\}_{n=1}^{\infty}$ may converge to a segment different from Γ because of the uniqueness of Γ. So, for n large enough, $\Gamma'_n \in \mathcal{N}$. Thus Γ'_n plus the subarc of Γ_1 from y_n to y_1 is shorter than Γ_1 and lies in $\mathcal{N}$. Hence y is conjugate to x.

Two sets $X, X' \in \mathcal{A}$ with a common point y will be called *orthogonal at* y if for any segments $xy, x'y \subset \mathcal{A}$ with $x \in X \setminus \{y\}$ and $x' \in X' \setminus \{y\}$, whenever x and x' converge to y, then the angle $\angle xyx'$ converges to $\pi/2$.

Theorem. *Let $\mathcal{A}$ be an Aleksandrov space, let $x \in \mathcal{A}$, let $y \in C(x)$ be a point closest to x among all points in some neighbourhood in $C(x)$ and let $\mathcal{F}$ be the space of all segments from x to y endowed with the Pompeiu-Hausdorff metric.*

If $\mathcal{F}$ is connected then y is conjugate to x. If $\mathcal{F}$ is disconnected then y is the midpoint of a closed geodesic arc at x dividing it into two segments; these are the only segments from x to y, and they are orthogonal to $C(X)$ at y.

Proof. Let $y \in C(x)$ be closest to x among all points of $C(x)$ from the open ball $B(y, \varepsilon) = \{y' \in \mathcal{A} : \rho(y, y') < \varepsilon\}$.

In view of Lemma 4, we may suppose that there are two segments Γ_1, Γ_2 from x to y. If they lie in the same component of $\mathcal{F}$, then obviously y is conjugate to x (there is a whole convergent sequence of distinct segments in $\mathcal{F}$). So, suppose Γ_1, Γ_2 lie in different components of $\mathcal{F}$.

If in some neighbourhood of y there are no points $z_1 \in \Gamma_1$, $z_2 \in \Gamma_2$ at a distance smaller than the length of $z_1y \cup yz_2$, then $\Gamma_1 \cup \Gamma_2$ is a closed geodesic arc at x, and y is obviously its midpoint. So, assume in each open ball $B(y, \delta)$ with $\delta < \varepsilon/2$ there are points z_1, z_2 on Γ_1, Γ_2 respectively, joined by an arc Φ shorter than $z_1y \cup yz_2$.

Suppose to Φ belongs a point p joined with x by at least two segments. Then $p \in C(x)$. Also, either z_1p is shorter than z_1y or z_2p is shorter than z_2y. So $p \in B(y, \varepsilon)$ and either $xz_1 \cup z_1p$ or $xz_2 \cup z_2p$ is shorter than Γ_1, which is impossible, because y is supposed to be a point of $C(x) \cap B(y, \varepsilon)$ closest to x. Hence every point $u \in \Phi$ is joined by a unique segment Γ_u with x, and let $\gamma_u \in \Sigma_x$ be its direction at x. The map $u \mapsto \Gamma_u$ is continuous.

Let $\pi_\Phi = \{\gamma_u : u \in \Phi\}$. When $z_1, z_2 \to y$ on Γ_1, Γ_2 the arc $\pi_\Phi \subset \Sigma_x$ keeps fixed its endpoints $\gamma_{z_1}, \gamma_{z_2}$. We may select sequences $z_1^{(n)}, z_2^{(n)} \to y$ such that, for the corresponding arcs $\Phi^{(n)}$, $\pi_{\Phi^{(n)}}$ converges to some arc π, say. Then $\gamma_{z_1}, \gamma_{z_2} \in \pi$ and in each direction from π we have a segment from x to y. But this shows that Γ_1, Γ_2 lie in the same component of $\mathcal{F}$, which contradicts our assumption.

Suppose now that there exists a third segment Γ_3 between x and y. Clearly, Γ_3 does not belong to at least one of the two components containing Γ_1 and Γ_2, say to the first one. Then $\Gamma_1 \cup \Gamma_3$ must also be a closed geodesic arc. Since geodesics do not branch, this is impossible. Hence Γ_1 and Γ_2 are the only segments from x to y.

Let now $x_1 \in \mathrm{int}\Gamma_1$, $x_2 \in \mathrm{int}\Gamma_2$ and let $y'_n \in C(x)$ converge to y for $n \to \infty$. If $\angle x_i y y'_n \to \alpha$ for some $\alpha < \pi/2$ then, by Lemma 2, $\angle^* x_i y y'_n < \alpha'$ for some $\alpha' \in (\alpha, \pi/2)$ and for n large enough. But this implies $\rho(x, y'_n) < \rho(x, y)$, which contradicts the assuption.

On the other hand, by Lemma 1, $\angle x_1 y y'_n + \angle x_2 y y'_n = \pi$, which implies $\angle x_1 y y'_n \to \pi/2$ for $n \to \infty$, because the existence of a subsequence of angles $\angle x_1 y y'_n$ convergent to a number larger than $\pi/2$ yields the existence of a subsequence of angles $\angle x_2 y y'_n$ converging to a number smaller than $\pi/2$, which was just shown to be impossible. Hence Γ_1 and Γ_2 are orthogonal to $C(x)$.

We remark that in arbitrary Aleksandrov spaces a point of $C(x)$ closest to x may not exist. In fact, by Lemma 3, this is the generic behaviour for the case of closed convex surfaces.

The Theorem obviously implies Theorem K and has the following Corollary, already mentioned by Kobayashi [4] in the frame of Riemannian manifolds.

Corollary. *If $(\mathcal{A}, \delta)$ is an Aleksandrov space and $\delta(x, y)$ is minimal among all $x \in \mathcal{A}$ and $y \in C(x)$, then either x and y are conjugate to each other, or x and y lie on a closed geodesic and determine on it two equally long segments.*

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