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The Dimension Print of Most Convex Surfaces

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Abstract. The dimension print is a concept which contains more detailed information than the usual Hausdorff dimension. So, for example, a sphere and the surface of a cube have same dimension but different dimension prints. Can anything be said about the dimension print of most convex surfaces (in the Baire category sense)?

Introduction

Most convex curves are differentiable like the circle, have many points with vanishing curvature like the square, are strictly convex like the circle, are not C^2 like the square, etc. Their dimension? It is of course 1, just as for both the circle and the square. It seems abstruse to try to discover a difference between them via dimension. It seemed so, until C. A. Rogers' dimension print appeared in print.

This new notion carries more information than the usual notion of dimension due to Hausdorff. For example, distinct Jordan curves may have quite different dimension prints. So the dimension print of the circle is triangular, while the square has a line-segment as dimension print.

What does the dimension print of most convex curves look like?

For the reader's convenience we recall here some definitions due to C. A. Rogers [4].

Let $\mathscr{B}$ denote the family of all boxes (i.e. rectangular parallelepipeds) in $\mathbb{R}^n$. For every box $B\in\mathscr{B}$ let $l_1(B),\ldots,l_n(B)$ be the edge lengths of B in non-increasing order (for pairwise non-parallel edges). Let $\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{R}^n$ be non-negative, i.e. $\alpha_i\geqslant 0$ $(1\leqslant i\leqslant n)$. For any set $S\subset\mathbb{R}^n$ and number $\delta>0$, let

$$\mu_\delta^\alpha(S)=\inf\left\{\sum_{i=1}^{\infty} l_1^{\alpha_1}(B_i)l_2^{\alpha_2}(B_i)\cdots l_n^{\alpha_n}(B_i): B_i\in\mathscr{B}, \operatorname{diam} B_i\leqslant\delta, \bigcup_{i=1}^{\infty} B_i\supset S\right\}.$$

Then

$$\mu^\alpha=\sup_{\delta>0}\mu_\delta^\alpha$$

is a measure on the space of all Borel sets in $\mathbb{R}^n$ and the *dimension print* of S is defined by

$$\operatorname{print} S=\{\alpha\in\mathbb{R}^n:\alpha \text{ is non-negative and } \mu^\alpha(S)>0\}.$$

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The *successive dimensions* $d_1(S), d_2(S), \dots, d_n(S)$ of S are defined by

$$d_i(S) = \sup\{\alpha_i : \mu^{(0,\dots,0,\alpha_i,0,\dots,0)}(S) > 0\} \quad (1 \leqslant i \leqslant n).$$

The first of them, $d_1(S)$, equals the usual Hausdorff dimension of S.

The boundary of an open bounded convex set in $\mathbb{R}^n$ is called a *convex surface* and the space $\mathscr{S}$ of all convex surfaces, endowed with the Pompeiu-Hausdorff metric, is a Baire space.

When we say that *most* elements of a Baire space have property **P** we mean that all elements except those in a set of first category enjoy property **P**.

For a survey of properties shared by most convex surfaces see [1], [5]. For information on Hausdorff measures and dimension see [3]. For details on the dimension print see [4].

We shall prove here that the answer to our question is: Like the square! In $\mathbb{R}^n$ the boundary of the hypercube, as well as every polytopal surface of dimension $n-1$, has the dimension print of a hyperplane and we shall prove that this is also the dimension print of most convex surfaces.

Auxiliary Results

Lemma 1. *Let $A \subset \mathbb{R}^n$ be a Borel set of successive dimensions $d_1, d_2, \dots, d_n$ and assume $d_{k+1} = 0$. Then*

$$\operatorname{print} A \subset \operatorname{print} H,$$

where H is a k-dimensional plane in $\mathbb{R}^n$.

Proof. By Lemma 5.2 in [4],

$$0 \leqslant d_i - d_{i+1} \leqslant 1 \quad (1 \leqslant i < n).$$

It follows that

$$d_i \leqslant k + 1 - i \quad (1 \leqslant i \leqslant k),$$
$$d_i = 0 \quad (i \geqslant k+1).$$

On the other hand, by Corollary 5.1 in [4], if $(\alpha_1, \dots, \alpha_n) \in \operatorname{print} A$ and $\beta_1, \dots, \beta_n \geqslant 0$ satisfy the inequalities

$$\beta_1 + \beta_2 + \cdots + \beta_n \leqslant \alpha_1 + \alpha_2 + \cdots + \alpha_n,$$
$$\beta_2 + \cdots + \beta_n \leqslant \alpha_2 + \cdots + \alpha_n,$$
$$\cdots\cdots\cdots\cdots$$
$$\beta_n \leqslant \alpha_n,$$

then $(\beta_1, \dots, \beta_n) \in \operatorname{print} A$.

Suppose now that $(\alpha_1, \dots, \alpha_n) \in \operatorname{print} A$. Then, for each i,

$$(0, \dots, 0, \alpha_i + \cdots + \alpha_n, 0, \dots, 0) \in \operatorname{print} A$$

too. Hence the inequalities

$$\alpha_i + \cdots + \alpha_n \leqslant d_i \leqslant k + 1 - i \quad (1 \leqslant i \leqslant k),$$
$$\alpha_i + \cdots + \alpha_n \leqslant d_i = 0 \quad (i \geqslant k+1)$$

characterizing the dimension print of H (see Example 1 in [4]) hold.

Lemma 2. *Suppose H is a hyperplane in $\mathbb{R}^n$ and $P \subset H$ is an arbitrary polytope. Also, let $s_1, \dots, s_{n-1} \geqslant 0$ and $t, \varepsilon, \delta > 0$. Then there exists an open set $P_0 \supset P$ and a finite family $\mathscr{B}^*$ of boxes with diameters at most ε, such that $\mathscr{B}^*$ be a covering of P_0 and*

$$\sum_{B \in \mathscr{B}^*} l_1(B)^{s_1} \cdots l_{n-1}(B)^{s_{n-1}} l_n(B)^t < \delta.$$

Proof. We may suppose that $H = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n = 0\}$. It suffices to take $P = [0, q]^{n-1} \times \{0\}$, because any other polytope in H is included in a translate of P for a suitable $q > 0$.

For $x \in H$ and $\rho > 0$, let $K(x, \rho)$ denote the hypercube in H with the centre at x, with edges parallel to the coordinate axes and with edge length 2ρ. Choose $p \in \mathbb{N}$ such that the diameter of the hypercube in $\mathbb{R}^n$ of edge length $2r$ be smaller than ε, where $r = q/p$. We consider the set

$$\Xi = \{(x_1, \dots, x_{n-1}, 0) \in P : x_j = kr, \quad 0 \leqslant k \leqslant p, \quad 1 \leqslant j \leqslant n-1\}$$

with $(p+1)^{n-1}$ elements. Choose $\eta \in (0, r)$ so that

$$(p+1)^{n-1} (2r)^{s_1 + s_2 + \cdots + s_{n-1}} \eta^t < \delta$$

and let P_0 be the set of all points in $\mathbb{R}^n$ at distance less than η from P.

Then P_0 is the desired open set including P and

$$\mathscr{B}^* = \{K(x, r) \times [-\eta, \eta] : x \in \Xi\}$$

is the desired covering of P_0.

Lemma 3. *If C is a convex surface and H a hyperplane in $\mathbb{R}^n$ then* $\operatorname{print} H \subset \operatorname{print} C$.

Proof. Let C_H be the orthogonal projection of C on H and consider an arbitrary covering $\mathscr{E}$ of C with ellipsoids of diameter at most δ. For each m-dimensional ellipsoid $E \subset \mathbb{R}^n$, let $a_1(E), a_2(E), \dots, a_m(E)$ be the lengths of its axes in non-increasing order. By Lemma 2.4 in [4], for the orthogonal projection E_H of $E \in \mathscr{E}$ on H the inequalities

$$a_j(E) \geqslant a_j(E_H) \quad (1 \leqslant j \leqslant n-1)$$

hold. Thus, for any numbers $s_1, \dots, s_{n-1} \geqslant 0$,

$$\sum_{E \in \mathscr{E}} a_1(E)^{s_1} a_2(E)^{s_2} \cdots a_{n-1}(E)^{s_{n-1}} \geqslant \sum_{E \in \mathscr{E}} a_1(E_H)^{s_1} a_2(E_H)^{s_2} \cdots a_{n-1}(E_H)^{s_{n-1}}.$$

By Theorem 3.1 in [4], if $(s_1, \dots, s_{n-1}, 0) \in \operatorname{print} C_H$ then the right side of the preceding inequality is positive for some $\delta > 0$, whence its left side is also positive and $(s_1, \dots, s_{n-1}, 0) \in \operatorname{print} C$.

Now, since C_H includes a hypercube in H (which has the same dimension print as H), $\operatorname{print} C_H = \operatorname{print} H$ and therefore $\operatorname{print} H \subset \operatorname{print} C$.

Main Result

We have now all we need to prove our main result.

Theorem. *Most convex surfaces in $\mathbb{R}^n$ have the dimension print of a hyperplane.*

Proof. Let $t>0$ and $m\in\mathbb{N}$. Denote by $\mathscr{F}_m$ the family of all convex surfaces not admitting any covering $\mathscr{B}^*$ with boxes of diameter at most $1/m$ such that

$$\sum_{B\in\mathscr{B}^*} l_n(B)^t < 1/m.$$

We shall prove that $\mathscr{F}_m$ is nowhere dense in $\mathscr{S}$.

Let $\mathscr{O}\subset\mathscr{S}$ be open and choose a polytope P with boundary $\operatorname{bd} P\in\mathscr{O}$. Let $F_1,\ldots,F_k$ denote the facets of P. By Lemma 2, we can find open sets $G_j\supset F_j$ and finite coverings $\mathscr{B}_j$ of G_j with boxes of diameter at most $1/m$ ($1\leqslant j\leqslant k$) such that

$$\sum_{B\in\mathscr{B}_j} l_n(B)^t < 1/km.$$

Then the open set

$$P_0=\bigcup_{j=1}^k G_j$$

includes $\operatorname{bd} P$ and, for the covering

$$\mathscr{B}^*=\bigcup_{j=1}^k \mathscr{B}_j$$

of P_0, the inequality

$$\sum_{B\in\mathscr{B}^*} l_n(B)^t < 1/m$$

holds. The surfaces in a whole neighbourhood $\mathscr{P}\subset\mathscr{O}$ of P lie in P_0 and admit the covering $\mathscr{B}^*$. Thus

$$\mathscr{P}\cap\mathscr{F}_m=\varnothing$$

and $\mathscr{F}_m$ is nowhere dense.

It follows that, for most $C\in\mathscr{S}$, we have $C\notin\bigcup_m\mathscr{F}_m$ and therefore

$$\mu_{1/m}^{(0,\ldots,0,t)}(C)\leqslant 1/m$$

for all m, which implies

$$\mu^{(0,\ldots,0,t)}(C)=0.$$

This leads to $d_n(C)=0$ which, by Lemma 1, implies $\operatorname{print} C\subset\operatorname{print} H$, where $H\subset\mathbb{R}^n$ is a hyperplane. This together with Lemma 3 proves the theorem.

Notice that every C^∞ convex curve in the plane has the dimension print of the circle (see [4], § 1) and therefore successive dimensions 1, 1/2, while most convex curves have, by our Theorem, the dimension print of the line, therefore successive dimensions 1, 0. Similarly, every C^∞ convex surface in $\mathbb{R}^3$ with nowhere vanishing curvature appears to have the dimension print of a sphere (see [4], § 1), with successive dimensions 2, 3/2, 1, while most convex surfaces have the dimension print of a plane and successive dimensions 2, 1, 0.

Combining a result of V. Klee [2] with our Theorem we see that most convex surfaces have the dimension print of a polytopal surface and are simultaneously C^1 and strictly convex.

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