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Convex Bodies Instead of Needles in Buffon's Experiment

A. ALEMAN[1], M. STOKA[2] and T. ZAMFIRESCU[3]*
[1]*Fachbereich Mathematik, Fernuniversität Hagen, 58084 Hagen, Germany.*
[2]*Dipartimento di Matematica, Università di Torino, 10123 Torino, Italy.*
[3]*Fachbereich Mathematik, Universität Dortmund, 44221 Dortmund, Germany.*
e-mail: tudor.zamfirescu@mathematik.uni-dortmund.de

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Abstract. In this paper we consider the two events that a random congruent copy of a convex body meets each one of two given families of equidistant lines in the plane. The probabilities are easily calculated. Then it is discovered that there always exists a value for the angle α between the nonparallel lines, such that the two events be independent. For convex bodies of constant width, and only for them, the two events are independent for any α.

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1. Introduction

The idea of repeating Buffon's experiment using other objects instead of a needle is not new. Various special planar convex bodies have been investigated in the literature; we shall consider here the general case. Of course, to go beyond convexity makes no sense, because only the convex cover is relevant to our problem (if the object is supposed connected).

Among the particular cases already treated in the literature we mention those of a circular disc [6], a sector thereof [4], a segment thereof [3], a (not necessarily symmetric) lens [3], and an ellipse [2].

Also, we follow the idea of considering not only one family of parallel lines but two, and of studying the hitting probability for the resulting lattice and the independency case of the two hitting events.

Consider a convex body (which means here a compact convex set) $K \subset \mathbb{R}$. Let $\mathcal{R}_a$ be a set of equidistant parallel lines in $\mathbb{R}$ (at distance a) and $\mathcal{R}_b$ another such set of lines (at distance b), the two directions making an angle $\alpha \in (0, \pi)$. The objects of our investigation are the events I_a, I_b that the random convex body K – more precisely the random congruent copy of K – meets some line in $\mathcal{R}_a$, $\mathcal{R}_b$ respectively.

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Let $L(\phi)$ be the width of K in direction ϕ. We consider the following natural condition.

$$\max_{0\leqslant\phi<\pi} L(\phi) < \min\{a, b\}. \tag{$*$}$$

All cells of the lattice $\mathcal{R}_a \cup \mathcal{R}_b$ are congruent to a parallelogram Π.

Let $\mathcal{K}$ be the set of all convex bodies congruent with K and with their centroids (just to make a choice) inside Π. We consider our convex bodies as uniformly distributed, in the sense that the centroid as a random variable is uniformly distributed in Π and the random variable ϕ (the rotation angle) is uniformly distributed in the interval $[0, 2\pi]$.

2. The Hitting Probability

Our goal here is the calculation of the probability $p = P(I_a \cup I_b)$.

Let B be the length of the boundary of K (counted twice if K is a line segment).

THEOREM 1. *If condition* $(*)$ *is satisfied then*

$$p = \frac{1}{\pi ab}\left(B(a+b) - \int_0^{\pi} L(\phi)L(\phi+\alpha)\,\mathrm{d}\phi\right).$$

Proof. If $\mathcal{L}$ is the set of those convex bodies congruent with K which are included in Π, then

$$p = 1 - \frac{\mu(\mathcal{L})}{\mu(\mathcal{K})},$$

where μ is the usual elementary kinematic measure in $\mathbb{R}^2$.

Let Π_ϕ be the set of the centroids of all translates of K rotated with the angle ϕ, included in Π.

Clearly,

$$\mu(\mathcal{K}) = \int_0^{2\pi} \mathrm{d}\phi \int_{(x,y)\in\Pi} \mathrm{d}x\,\mathrm{d}y = \frac{2\pi ab}{\sin\alpha.}$$

Since the sides of Π_ϕ have lengths

$$\frac{a - L(\phi)}{\sin\alpha}, \qquad \frac{b - L(\phi+\alpha)}{\sin\alpha},$$

we have

$$\mu(\mathcal{L}) = \int_0^{2\pi} \mathrm{d}\phi \int_{(x,y)\in\Pi_\phi} \mathrm{d}x\,\mathrm{d}y = \int_0^{2\pi} \frac{(a - L(\phi))(b - L(\phi+\alpha))}{\sin\alpha}\,\mathrm{d}\phi$$

$$= \frac{2\pi ab}{\sin\alpha} - \frac{a}{\sin\alpha}\int_0^{2\pi} L(\phi+\alpha)\,\mathrm{d}\phi -$$

$$-\frac{b}{\sin\alpha}\int_0^{2\pi} L(\phi)\,\mathrm{d}\phi + \frac{1}{\sin\alpha}\int_0^{2\pi} L(\phi)L(\phi+\alpha)\,\mathrm{d}\phi.$$

Hence

$$p = \frac{1}{2\pi ab}\left(a\int_0^{2\pi} L(\phi+\alpha)\,\mathrm{d}\phi + b\int_0^{2\pi} L(\phi)\,\mathrm{d}\phi - \right.$$

$$\left. - \int_0^{2\pi} L(\phi)L(\phi+\alpha)\,\mathrm{d}\phi\right).$$

Since $L(\phi) = L(\phi+\pi)$ and

$$\int_0^{2\pi} L(\phi)\,\mathrm{d}\phi = \int_0^{2\pi} L(\phi+\alpha)\,\mathrm{d}\phi = 2B,$$

we get

$$p = \frac{1}{\pi ab}\left(B(a+b) - \int_0^{\pi} L(\phi)L(\phi+\alpha)\,\mathrm{d}\phi\right)$$

and the theorem is proved.

Remarks. By letting $b \to \infty$ in Theorem 1, we find

$$P(I_a) = \frac{B}{\pi a},$$

the probability that the convex body K with $L(\phi) < a$ for every ϕ meets $\mathcal{R}_a$, result discovered by Barbier [1] not quite recently, namely in 1860.

For a convex body K of constant width k smaller than both a and b,

$$p = \frac{1}{\pi ab}(B(a+b) - \pi k^2) = \frac{k}{ab}(a+b+k).$$

This extends the formula known for a circular disc of diameter k ([7, p. 42]).

In the case of Buffon's experiment, as K is a line segment of length l with $L(\phi) = l|\sin\phi|$, Theorem 1 gives

$$p = \frac{1}{\pi ab}[2l(a+b) - (\sin\alpha + (\tfrac{1}{2}\pi - \alpha)\cos\alpha)l^2],$$

which verifies a result from ([6, p. 55]).

For K an ellipse of half-axes ξ, ζ, the width is $L(\phi) = 2\sqrt{\xi^2 \sin^2 \phi + \zeta^2 \cos^2 \phi}$ and, by Theorem 1,

$$p = \frac{1}{\pi ab}\Bigg(B(a+b) - \\ -4\int_0^{\pi} \sqrt{(\xi^2 \sin^2 \phi + \zeta^2 \cos^2 \phi)(\xi^2 \sin^2(\phi+\alpha) + \zeta^2 + \cos^2(\phi+\alpha))}\,\mathrm{d}\phi\Bigg),$$

which extends a formula known for the case $\alpha = \pi/2$ (see [2, p. 971]).

3. Auxiliary Material

In the last section we intend to treat the case of independency of the events I_a and I_b. This section will provide the necessary technical basis.

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a periodic function, of period $2\pi/m$, with $m \in \mathbb{N}$, such that $f|_{[0,2\pi]} \in L^2([0, 2\pi])$.

PROPOSITION. *Consider the equality*

$$\frac{1}{2\pi}\int_0^{2\pi} f(\phi)f(\phi+\alpha)\,\mathrm{d}\phi = \left(\frac{1}{2\pi}\int_0^{2\pi} f(\phi)\,\mathrm{d}\phi\right)^2. \tag{$**$}$$

(a) *As an equation in* α, $(**)$ *has at least one solution in* $[0, \pi/m]$.
(b) *The equation* $(**)$ *has the solution* $\alpha = 0$ *if and only if f is constant.*

Proof. Let $\Sigma_{-\infty}^{\infty} a_n\,\mathrm{e}^{in\phi}$ be the Fourier series of f. Then the Fourier series of the function $f(\phi+\alpha)$ with the variable ϕ is $\Sigma_{-\infty}^{\infty} a_n\,\mathrm{e}^{in\alpha}\,\mathrm{e}^{in\phi}$, and

$$f|_{[0,2\pi]} \in L^2([0, 2\pi]) \text{ if and only if } \sum_{-\infty}^{\infty} |a_n|^2 < \infty, \tag{1}$$

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} f(\phi)f(\phi+\alpha)\,\mathrm{d}\phi &= \frac{1}{2\pi}\int_0^{2\pi} \overline{f(\phi)}f(\phi+\alpha)\,\mathrm{d}\phi \\ &= \sum_{-\infty}^{\infty} |a_n|^2\,\mathrm{e}^{in\alpha}.\end{aligned} \tag{2}$$

Moreover, we have

$$\int_0^{2\pi} f(\phi)f\left(\phi + \frac{2\pi}{m} - \alpha\right)\mathrm{d}\phi = \int_0^{2\pi} f(\phi)f(\phi+\alpha)\,\mathrm{d}\phi. \tag{3}$$

Indeed, the periodicity of f yields

$$\int_0^{2\pi} f(\phi) f\left(\phi + \frac{2\pi}{m} - \alpha\right) \mathrm{d}\phi$$

$$= \int_0^{2\pi} f(\phi) f(\phi - \alpha)\, \mathrm{d}\phi = \int_{-\alpha}^{2\pi-\alpha} f(t+\alpha) f(t)\, \mathrm{d}t$$

$$= \int_{-\alpha}^{0} f(t+\alpha) f(t)\, \mathrm{d}t + \int_0^{2\pi-\alpha} f(t+\alpha) f(t)\, \mathrm{d}t.$$

As a function of t, $f(t+\alpha)f(t)$ has period 2π, whence

$$\int_{-\alpha}^{0} f(t+\alpha) f(t)\, \mathrm{d}t = \int_{2\pi-\alpha}^{2\pi} f(t+\alpha) f(t)\, \mathrm{d}t$$

and (3) follows.

Thus, $(**)$ becomes

$$\sum_{-\infty}^{\infty} |a_n|^2 \mathrm{e}^{in\alpha} = |a_0|^2. \tag{$**'$}$$

Now (b) follows immediately, because for $\alpha = 0$ $(**')$ is equivalent to $|a_n|^2 = 0$ for all $n \neq 0$, i.e. $f = a_0$.

From (1) and (2) it follows that the function

$$u(\mathrm{e}^{i\alpha}) = \frac{1}{2\pi} \int_0^{2\pi} f(\phi) f(\phi + \alpha)\, \mathrm{d}\phi = \sum_{-\infty}^{\infty} |a_n|^2\, \mathrm{e}^{in\alpha}$$

is continuous, and its Fourier series, whose coefficients are $|a_n|^2$, converges uniformly to u. From (3) it follows that

$$u(\mathrm{e}^{i\alpha}) = u(\mathrm{e}^{i((2\pi/m)-\alpha)}) \tag{4}$$

and from the periodicity of f it follows that

$$u(\mathrm{e}^{i(\alpha+(2\pi/m))}) = u(\mathrm{e}^{i\alpha}). \tag{5}$$

To show (a), let $\tilde{u}$ be the harmonic function in $D = \{z \in \mathbb{C}\colon |z| < 1\}$, continuous in $\overline{D}$, with $\tilde{u}(\mathrm{e}^{i\alpha}) = u(\mathrm{e}^{i\alpha})$ (the solution of Dirichlet's problem with boundary data u). Then

$$\tilde{u}(0) = \frac{1}{2\pi} \int_0^{2\pi} \tilde{u}(\mathrm{e}^{i\alpha})\, \mathrm{d}\alpha = \frac{1}{2\pi} \int_0^{2\pi} u(\mathrm{e}^{i\alpha})\, \mathrm{d}\alpha = |a_0|^2.$$

Since $\tilde{u}$ is continuous in $\overline{D}$, it attains its absolute maximum and minimum on the boundary, whence there are points $\alpha_1, \alpha_2 \in [0, 2\pi]$ with

$$\tilde{u}(\mathrm{e}^{i\alpha_1}) \leq \tilde{u}(0) \leq \tilde{u}(\mathrm{e}^{i\alpha_2}),$$

in other words

$$u(\mathrm{e}^{i\alpha_1}) \leq |a_0|^2 \leq u(\mathrm{e}^{i\alpha_2}).$$

From the continuity of u it follows that $u(\mathrm{e}^{i\alpha_0}) = |a_0|^2$ for some $\alpha_0 \in [0, 2\pi]$. Then

$$u(\mathrm{e}^{i\alpha_0}) = \sum_{-\infty}^{\infty} |a_n|^2 \mathrm{e}^{in\alpha_0} = |a_0|^2,$$

whence α_0 is a solution of $(**')$.

From (5) it follows that $(**')$ has a solution in $[0, 2\pi/m]$, and from (4) we see that there is a solution even in $[0, \pi/m]$.

Remark. As the above argument shows, the left-hand side of $(**)$ is a continuous function of α. Then, by (b), it follows that 0 cannot even be a limit point of solutions of $(**)$ unless f is constant. Hence $(**)$ holds for arbitrarily small values of α if and only if f is constant.

4. The Independency Case

When are the events I_a and I_b independent? The characterization of this case is the aim of the next theorem.

THEOREM 2. *Suppose condition $(*)$ is satisfied. Then the events I_a and I_b are independent if and only if*

$$\int_0^{\pi} L(\phi)L(\phi + \alpha)\,\mathrm{d}\phi = \frac{B^2}{\pi}.$$

Proof. The probability of I_a, I_b happening simultaneously is

$$P(I_a \cap I_b) = P(I_a) + P(I_b) - P(I_a \cup I_b),$$

where $P(I_a) = B/(\pi a)$, $P(I_b) = B/(\pi b)$ and p was computed in Theorem 1. Thus,

$$P(I_a \cap I_b) = \frac{1}{\pi ab} \int_0^{\pi} L(\phi)L(\phi + \alpha)\,\mathrm{d}\phi.$$

The events I_a and I_b are independent precisely when $P(I_a \cap I_b) = P(I_a)P(I_b)$, which yields the condition from the statement.

Remarks. The condition in Theorem 2 is verified for all angles α if K has constant width. Hence, for such a K and any α, I_a and I_b are independent. For the case of a circular disc this was known (see [7, p. 43]). Is the condition in Theorem 2 verified for all α only if K has constant width? The answer is provided by the next result.

THEOREM 3. *If K has constant width then the events I_a and I_b are independent for any angle α. If I_a and I_b are independent for arbitrarily small angles α then K has constant width.*

Proof. The width L of a convex body is periodic with period π. Now, taking $f = L$ and $m = 2$, the theorem follows from the remark in the preceding section.

In the case of Buffon's needle, when K is a line segment of length l, the condition from Theorem 2 becomes

$$l^2 \int_0^\pi |\sin\phi \sin(\phi+\alpha)| \, \mathrm{d}\phi = \frac{4l^2}{\pi},$$

which is equivalent with

$$\int_0^{\pi-\alpha} \sin\phi \sin(\phi+\alpha) \, \mathrm{d}\phi - \int_{\pi-\alpha}^{\pi} \sin\phi \sin(\phi+\alpha) \, \mathrm{d}\phi = \frac{4}{\pi}.$$

This leads directly to the condition

$$\sin\alpha + \left(\frac{\pi}{2} - \alpha\right)\cos\alpha = \frac{4}{\pi},$$

which is fulfilled by a single certain angle α, first discovered by Schuster [5] in 1974.

Again, a natural question arises. Is the case of Buffon's needle singular or not?

THEOREM 4. *For any convex body K there is a nonvanishing angle α for which I_a and I_b are independent.*

Proof. If K has constant width then I_a and I_b are independent for all α, by Theorem 3. Otherwise, there is an α for which I_a and I_b are independent by the Proposition, part (a), and $\alpha \neq 0$ by its part (b).

For the reader's pleasure we finish the paper with the following open problem.

PROBLEM. Characterize the convex bodies K, such that the angle α, for which I_a and I_b are independent, be unique.

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