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Cardinality of the metric projection on typical compact sets in Hilbert spaces

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1. *Introduction*

The metric projection mapping π_X plays an important role in nonlinear approximation theory. Usually X is a closed subset of a Banach space $\mathbb{E}$ and, for each $e \in \mathbb{E}$, $\pi_X(e)$ is the set, perhaps empty, of all points in X which are nearest to e. From a classical theorem due to Stečkin [**7**] it is known that, when $\mathbb{E}$ is uniformly convex, the metric projection $\pi_X(e)$ is single valued at each typical point e of $\mathbb{E}$ (in the sense of the Baire categories), i.e. at each point e of a residual subset of $\mathbb{E}$. More recently Zamfirescu [**8**] has proven that, if X is a typical compact set in $\mathbb{R}^n$ (in the sense of Baire categories) and $n \geqslant 2$, then the metric projection $\pi_X(e)$ has cardinality at least 2 at each point e of a dense subset of $\mathbb{R}^n$. This result has been extended in several directions by Zhivkov [**9**, **10**], who has also considered the case of the metric antiprojection mapping ν_X (which associates with each $e \in \mathbb{E}$ the set $\nu_X(e)$, perhaps empty, of all $\in X$ which are farthest from e). For this mapping De Blasi [**2**] has shown that, if $\mathbb{E}$ is a real separable Hilbert space with $\dim \mathbb{E} = +\infty$ and n is an arbitrary natural number not less than 2, then, for a typical compact convex set $X \subset \mathbb{E}$, the metric antiprojection $\nu_X(e)$ has cardinality at least n at each point e of a dense subset of $\mathbb{E}$. A systematic discussion of the properties of the maps π_X and ν_X, and additional bibliography, can be found in Singer [**5**, **6**] and Dontchev and Zolezzi [**3**].

In the present paper we consider some further properties of the metric projection mapping π_X, with X a compact set in a real separable Hilbert space $\mathbb{E}$. If $\dim \mathbb{E} = n$ and $2 \leqslant n < +\infty$, it is proven that for a typical compact set $X \subset \mathbb{E}$, the metric projection $\pi_X(e)$ has cardinality exactly $n+1$ at each point e of a dense subset of $\mathbb{E}$, while the set of those points $e \in \mathbb{E}$ where $\pi_X(e)$ has cardinality at least $n+2$ is empty. Furthermore it is shown that, if $\dim \mathbb{E} = +\infty$, then for a typical compact set $X \subset \mathbb{E}$ the metric projection $\pi_X(e)$ has cardinality at least n (for arbitrary $n \geqslant 2$) at each point e of a dense subset of $\mathbb{E}$. Incidentally we obtain a characterization of the dimension of the space $\mathbb{E}$ by means of a typical property holding in the space of the compact subsets of $\mathbb{E}$.

2. *Notation and auxiliary results*

Throughout this paper $\mathbb{E}$ denotes a real Hilbert space with $\dim \mathbb{E} \geqslant 2$, with inner product $\langle .,. \rangle$, and induced norm $\|.\|$. $\mathscr{K}_{\mathbb{E}}$ stands for the space of the nonempty compact subsets of $\mathbb{E}$, endowed with the Pompeiu–Hausdorff metric h. As is well known, under the metric h the space $\mathscr{K}_{\mathbb{E}}$ is complete.

For $X \in \mathscr{K}_{\mathbb{E}}$ and $e \in \mathbb{E}$ we denote by $\pi_X(e)$ the metric projection of e on X, that is

$$\pi_X(e) = \{x \in X \mid \|x-e\| = d(X,e)\}, \tag{2·1}$$

where $d(X,e) = \min\{\|x-e\| \mid x \in X\}$. The map $\pi_X : \mathbb{E} \to \mathscr{K}_{\mathbb{E}}$ defined by (2·1) is called metric projection of $\mathbb{E}$ on X.

Let $X \in \mathscr{K}_{\mathbb{E}}$ and $n \in \mathbb{N}$, $n \geqslant 2$, be arbitrary. The sets

$$M^n(X) = \{e \in \mathbb{E} \mid \operatorname{card} \pi_X(e) = n\}$$

$$M^n_+(X) = \{e \in \mathbb{E} \mid \operatorname{card} \pi_X(e) \geqslant n\}$$

are called, respectively, the n-valued locus of π_X, and the n^+-valued locus of π_X.

A set X in a complete metric space M is called residual in M, if $M \setminus X$ is of the first Baire category in M. Elements of M enjoying a property shared by all elements of a set residual in M are said to be typical.

By $U_M(x,r)$, $\tilde{U}_M(x,r)$ we mean the open, respectively closed, ball in M with centre x and radius r.

As usual, $\mathbb{N}$ stands for the set of integers $n \geqslant 1$, and Q^+ for the set of the strictly positive rationals.

In the sequel we shall use the following topological result contained in an implicit form in Brouwer [**1**] which, as shown by Miranda [**4**], is equivalent to Brouwer's fixed point theorem.

Brouwer–Miranda Theorem. *Let* $Q^\theta_a = [a_1-\theta,\ a_1+\theta] \times \cdots \times [a_n-\theta,\ a_n+\theta]$, $\theta > 0$, *be a nondegenerate interval of* $\mathbb{R}^n$ *with centre* $a = (a_1, \dots, a_n)$ *and, for* $k = 1, \dots, n$ *let* $L^{\pm\theta}_k = \{(x_1, \dots, x_n) \in Q^\theta_a \mid x_k = a_k \pm \theta\}$. *Let* $f_k : Q^\theta_a \to \mathbb{R}$, $k = 1, \dots, n$, *be* n *continuous functions defined on* Q^θ_a *such that*:

$$f_k(x) < 0 \textit{ for every } x \in L^{-\theta}_k, \quad f_k(x) > 0 \textit{ for every } x \in L^{+\theta}_k,$$

where $x = (x_1, \dots, x_n)$. *Then there exists a point* $\hat{x} \in Q^\theta_a$ *such that* $f_k(\hat{x}) = 0$ *for* $k = 1, \dots, n$.

The theorem above remains valid also when Q^θ_a is a bounded polyhedron of the form $\{x \in \mathbb{R}^n \mid |\langle x-a, u_k\rangle| \leqslant \theta, k = 1, \dots, n\}$, where $u_1, \dots, u_n$ are n linearly independent vectors of $\mathbb{R}^n$, and accordingly $L^{\pm\theta}_k = \{x \in Q^\theta_a \mid \langle x-a, u_k\rangle = \pm\theta\}$, $k = 1, \dots, n$.

Proposition 1. *Suppose* $\dim \mathbb{E} = n \geqslant 2$. *Let* $A_0 \in \mathscr{K}_{\mathbb{E}}$, $e_0 \in \mathbb{E}$, $\lambda > 0$ *and* $r > 0$ *be arbitrary. Then there exist* $B \in \mathscr{K}_{\mathbb{E}}$ *and* $\sigma > 0$, *with* $U_{\mathscr{K}_{\mathbb{E}}}(B,\sigma) \subset U_{\mathscr{K}_{\mathbb{E}}}(A_0,\lambda)$, *such that for every* $X \in U_{\mathscr{K}_{\mathbb{E}}}(B,\sigma)$ *we have*

$$M^{n+1}_+(X) \cap U_{\mathbb{E}}(e_0, r) \neq \varnothing.$$

Proof. We adapt an argument from [**2**]. First we consider the case $d(A_0, e_0) > 0$.

Step 1. Construction of B.
Take $a_0 \in A_0$ so that $\|a_0 - e_0\| = d(A_0, e_0)$ and $\gamma > 0$, $\beta > 0$ satisfying

$$1 - \frac{\lambda}{4\|a_0 - e_0\|} < \gamma < 1, \quad \gamma - \frac{\lambda^2}{32\|a_0 - e_0\|^2} < \beta < \gamma.$$

Let $u_1, \ldots, u_{n-1}$ be $n-1$ mutually orthogonal vectors of norm 1 contained in the hyperplane $\{x \in \mathbb{E} \mid \langle x, a_0 - e_0 \rangle = 0\}$. Let $b_0, b_1, \ldots, b_n$ be given by

$$\begin{aligned} b_0 &= e_0 + \gamma(a_0 - e_0) \\ b_k &= e_0 + \beta(a_0 - e_0 + v_k, \quad k = 1, \ldots, n-1 \\ b_n &= e_0 + \beta(a_0 - e_0) - v_1, \end{aligned}$$

where $v_k = \sqrt{\gamma^2 - \beta^2}\, \|a_0 - e_0\|\, u_k$, and set

$$B = \{b_0, b_1, \ldots, b_n\} \cup A_0. \tag{2·2}$$

Observe that the vectors $b_1 - b_0, \ldots, b_n - b_0$ are linearly independent. Furthermore, we have:

$$\begin{aligned} \|b_k - b_0\| &= \sqrt{2\gamma(\gamma - \beta)}\, |a_0 - e_0\|, \;\; k = 1, \ldots, n \\ \|b_k - e_0\| &= \gamma\, \|a_0 - e_0\|, \qquad\qquad k = 0, 1, \ldots, n. \end{aligned} \tag{2·3}$$

The latter equality shows that the set $\{b_k\}_{k=0}^n$ lies on the boundary of the ball $\tilde{U}_{\mathbb{E}}(e_0, \gamma\, \|a_0 - e_0\|)$. Since $d(A_0, e_0) > \gamma\, \|a_0 - e_0\|$ and the b_ks are pairwise different we have

$$\pi_B(e_0) = \{b_0, b_1, \ldots, b_n\},$$

where $\operatorname{card} \pi_B(e_0) = n+1$. As $\gamma < 1$ and $\gamma - \beta < \lambda^2/(32\, \|a_0 - e_0\|^2)$, (2·3) gives $\|b_k - b_0\| < \lambda/4$, $k = 1, \ldots, n$. Further, $\|b_0 - a_0\| < \lambda/4$, for $\|b_0 - a_0\| = (1-\gamma)\, \|a_0 - e_0\|$ and $1 - \gamma < \lambda/(4\, \|a_0 - e_0\|)$. Hence, by the triangle inequality, $\|b_k - a_0\| < \lambda/2$, $k = 0, 1, \ldots, n$, which implies

$$h(B, A_0) < \lambda/2.$$

Now fix η satisfying

$$0 < \eta < \min\left\{r, \frac{\|b_0 - a_0\|}{4}\right\} \cup \left\{\frac{\|b_h - b_k\|}{4} \,\middle|\, h, k = 0, 1, \ldots, n, \quad h \neq k\right\} \tag{2·4}$$

and observe that

$$\tilde{U}_{\mathbb{E}}(b_h, 2\eta) \cap \tilde{U}_{\mathbb{E}}(b_k, 2\eta) = \varnothing, \quad h, k = 0, 1, \ldots, n, \quad h \neq k \tag{2·5}$$

$$\tilde{U}_{\mathbb{E}}(e_0, \|b_0 - e_0\| + 2\eta) \cap (\mathbb{E} \setminus U_{\mathbb{E}}(e_0, \|a_0 - e_0\| - 2\eta)) = \varnothing. \tag{2·6}$$

Furthermore, for $\theta > 0$ put:

$$\begin{aligned} Q_{e_0}^{\theta} &= \{x \in \mathbb{R}^n \mid |\langle x - e_0, b_k - b_0 \rangle| \leqslant \theta, \quad k = 1, \ldots, n\} \\ L_k^{\pm\theta} &= \{x \in Q_{e_0}^{\theta} \mid \langle x - e_0, b_k - b_0 \rangle = \pm\theta\}, \quad k = 1, \ldots, n. \end{aligned}$$

The polyhedron $Q_{e_0}^{\theta}$ is a bounded neighbourhood of e_0, whose diameter vanishes as $\theta \to 0$. Thus there is a $\theta > 0$ such that

$$Q_{e_0}^{\theta} \subset U_{\mathbb{E}}(e_0, \eta). \tag{2·7}$$

For $X \in U_{\mathscr{K}_{\mathbb{E}}}(B, \eta)$ we put:

$$X_k = X \cap \tilde{U}_{\mathbb{E}}(b_k, \eta), \quad k = 0, 1, \ldots, n, \quad \tilde{X} = X \cap (A_0 + \tilde{U}_{\mathbb{E}}(0, \eta)). \tag{2·8}$$

It is easy to verify, by (2·5) and (2·6), that X_0, $X_1, \ldots, X_n$, $\tilde{X}$ are pairwise disjoint nonempty compact sets, with

$$X_0 \cup X_1 \cup \cdots \cup X_n \cup \tilde{X} = X. \tag{2·9}$$

Step 2. Let η and θ satisfy (2·4) and (2·7). Then there is a σ with

$$0 < \sigma < \min\left\{\eta, \frac{\lambda}{2}\right\} \tag{2·10}$$

such that for every $X \in U_{\mathscr{K}_{\mathbb{E}}}(B,\sigma)$ and $k = 1, \ldots, n$ we have:

$$d(X_0, e) - d(X_k, e) < 0 \quad \text{for every } e \in L_k^{-\theta} \tag{2·11}$$

$$d(X_0, e) - d(X_k, e) > 0 \quad \text{for every } e \in L_k^{+\theta}, \tag{2·12}$$

where the X_ks are given by (2·8).

The X_ks are certainly nonempty and compact, with the above properties, since $\sigma < \eta$. Let $1 \leqslant k \leqslant n$. First we prove (2·11), with $X = B$. Clearly $B_0 = \{b_0\}$ and $B_k = \{b_k\}$; thus, for each $e \in L_k^{-\theta}$, we have:

$$\begin{aligned} d^2(B_0, e) - d^2(B_k, e) &= \|(b_0 - e_0) - (e - e_0)\|^2 - \|(b_k - e_0) - (e - e_0)\|^2 \\ &= \|b_0 - e_0\|^2 + \|e - e_0\|^2 - 2\langle b_0 - e_0, e - e_0\rangle - \|b_k - e_0\|^2 - \|e - e_0\|^2 \\ &\quad + 2\langle b_k - e_0, e - e_0\rangle \\ &= 2\langle b_k - b_0, e - e_0\rangle = -2\theta, \end{aligned}$$

and (2·11) is satisfied, with B in the place of X. The proof of (2·12), with B in the place of X, is similar. Since the multifunctions $X \mapsto X_k$, $k = 0, 1, \ldots, n$, are continuous and the sets $L_k^{\pm\theta}$, $k = 1, \ldots, n$, are compact, there is a σ satisfying (2·10), such that for every $X \in U_{\mathscr{K}_{\mathbb{E}}}(B,\sigma)$, (2·11) and (2·12) are fulfilled.

Step 3. With B given by (2·2) and σ defined in Step 2, the statement of Proposition 1 is true.

Clearly $U_{\mathscr{K}_{\mathbb{E}}}(B,\sigma) \subset U_{\mathscr{K}_{\mathbb{E}}}(A_0,\lambda)$, for $h(B, A_0) < \lambda/2$ and $\sigma < \lambda/2$. Let $X \in U_{\mathscr{K}_{\mathbb{E}}}(B,\sigma)$ be arbitrary. Then we have

$$M_+^{n+1}(X) \cap U_{\mathbb{E}}(e_0, r) \neq \varnothing. \tag{2·13}$$

In fact, by Step 2, the n continuous functions $e \mapsto d(X_0, e) - d(X_k, e)$, $k = 1, \ldots, n$ defined on the polyhedron $Q_{e_0}^{\theta}$ satisfy (2·11) and (2·12). By the Brouwer–Miranda theorem, there is a point $\hat{e} \in Q_{e_0}^{\theta}$ in which all of them vanish simultaneously. Thus

$$d(X_0, \hat{e}) = d(X_k, \hat{e}), \quad k = 1, \ldots, n. \tag{2·14}$$

We claim that $\hat{e} \in M_+^{n+1}(X) \cap U_{\mathbb{E}}(e_0, r)$. In fact, in view of (2·7) and of the definition of X_k and $\tilde{X}$, we have:

$$d(X_k, \hat{e}) \leqslant d(X_k, e_0) + \|\hat{e} - e_0\| < \|b_k - e_0\| + 2\eta = \|b_0 - e_0\| + 2\eta, \quad k = 0, 1, \ldots, n,$$

and

$$d(\tilde{X}, \hat{e}) \geqslant d(\tilde{X}, e_0) - \|\hat{e} - e_0\| > \|a_0 - e_0\| - 2\eta.$$

As $\eta < \|a_0 - b_0\|/4$, it follows that $d(X_k, \hat{e}) < d(\tilde{X}, \hat{e})$, $k = 0, 1, \ldots, n$. Combining the latter inequality with (2·9) and (2·14) gives

$$d(X, \hat{e}) = d(X_k, \hat{e}), \quad k = 0, 1, \ldots, n.$$

It follows that

$$X_k \cap \pi_X(\hat{e}) \neq \varnothing, \quad k = 0, 1, \ldots, n,$$

whence in each of the $n+1$ balls $\tilde{U}_{\mathbb{E}}(b_k, \eta)$ there is at least one point of $\pi_X(\hat{e})$. Since these balls are pairwise disjoint, we have $\operatorname{card}\pi_X(\hat{e}) \geqslant n+1$. Hence $\hat{e}\in M_+^{n+1}(X)$. Furthermore $\hat{e}\in U_{\mathbb{E}}(e_0, r)$, for $\hat{e}\in Q_{e_0}^{\theta} \subset U_{\mathbb{E}}(e_0, \eta)$ and $\eta < r$. Consequently (2·13) is verified and, under the assumption $d(A_0, e_0) > 0$, Proposition 1 is proved.

Now suppose $d(A_0, e_0) = 0$. In this case take $\tilde{A}\in\mathscr{K}_{\mathbb{E}}$ such that $d(\tilde{A}, e_0) > 0$ and $h(\tilde{A}, A_0) < \lambda/2$. Then there exist $B\in\mathscr{K}_{\mathbb{E}}$ and $\sigma > 0$, with $U_{\mathscr{K}_{\mathbb{E}}}(B, \sigma) \subset U_{\mathscr{K}_{\mathbb{E}}}(\tilde{A}, \lambda/2)$, such that each $X\in U_{\mathscr{K}_{\mathbb{E}}}(B, \sigma)$ satisfies (2·13). As $U_{\mathscr{K}_{\mathbb{E}}}(B, \sigma) \subset U_{\mathscr{K}_{\mathbb{E}}}(A_0, \lambda)$, the statement of Proposition 1 is satisfied. This completes the proof.

The following proposition can be proved as in [**2**] (see Lemma 4·1).

Proposition 2. *Let* $\mathbb{E}$ *be a real Hilbert space with* $\dim\mathbb{E} = +\infty$. *Let* $A_0\in\mathscr{K}_{\mathbb{E}}$, $e_0\in\mathbb{E}$, $\lambda > 0$, $r > 0$ *and* $n\in\mathbb{N}$ *be arbitrary. Then there exist* $B\in\mathscr{K}_{\mathbb{E}}$ *and* $\sigma > 0$, *with* $U_{\mathscr{K}_{\mathbb{E}}}(B, \sigma) \subset U_{\mathscr{K}_{\mathbb{E}}}(A_0, \lambda)$, *such that for every* $X\in U_{\mathscr{K}_{\mathbb{E}}}(B, \sigma)$ *we have*

$$M_+^{n+1}(X) \cap U_{\mathbb{E}}(e_0, r) \neq 0.$$

Now suppose that $\dim\mathbb{E} = n \geqslant 2$. Set $\mathscr{P}_m = \{X\in\mathscr{K}_{\mathbb{E}} \mid \operatorname{card} X = m\}$, $m\in\mathbb{N}$. For $X\in\mathscr{P}_m$, $X = \{x_1, \ldots, x_m\}$, define

$$m(X) = \min\{\|x_i - x_j\| \mid i, j = 1, \ldots, m, \quad i \neq j\}.$$

Given $n+1$ points $c_1, \ldots, c_{n+1}\in\mathbb{E}$, we denote by $S_{c_1\ldots c_{n+1}}$ any sphere in $\mathbb{E}$ containing $c_1, \ldots, c_{n+1}$. Observe that $S_{c_1\ldots c_{n+1}}$ exists and is unique if and only if for some (and so for each) r, $1 \leqslant r \leqslant n+1$, the set $\{c_k - c_r \mid k = 1, \ldots, n+1;\ k \neq r\}$ is linearly independent.

Proposition 3. *Suppose* k, $n\in\mathbb{N}$ *and* $\dim\mathbb{E} = n \geqslant 2$. *Then for every* $A\in\mathscr{P}_{n+k}$, $A = \{a_1, \ldots, a_{n+k}\}$, *and* $\epsilon > 0$, *there exists* $B\in\mathscr{P}_{n+k}$, $B = \{b_1, \ldots, b_{n+k}\}$, *with* $h(B, A) < \epsilon$, *such that for each set* $\{b_{i_1}, \ldots, b_{i_{n+1}}\}$ *of* $n+1$ *different points* $b_{i_r}\in B$ *the following two properties are satisfied*:

(i) *there is one and only one sphere* $S_{b_{i_1},\ldots,b_{i_{n+1}}}$ *containing* $b_{i_1}, \ldots, b_{i_{n+1}}$;

(ii) $S_{b_{i_1},\ldots,b_{i_{n+1}}} \cap B = \{b_{i_1}, \ldots, b_{i_{n+1}}\}$.

Proof. Denote by (P_k), $k\in\mathbb{N}$, the statement of the proposition. Clearly (P_1) holds true. The proof of (P_k), for any $k\in\mathbb{N}$, can be easily established by an induction argument.

3. *Main results*

In this section we study the cardinality of the metric projection mapping π_X for typical $X\in\mathscr{K}_{\mathbb{E}}$.

Theorem 1. *Let* $\mathbb{E}$ *be a real separable Hilbert and suppose either* $\dim\mathbb{E} = n \geqslant 2$, *or* $\dim\mathbb{E} = +\infty$ *and* $n\in\mathbb{N}$ *arbitrary. Then, for a typical* $X\in\mathscr{K}_{\mathbb{E}}$, *the* $(n+1)^+$*-valued locus of* π_X *is dense in* $\mathbb{E}$.

Proof. Let $E_0 \subset \mathbb{E}$ be countable and dense in $\mathbb{E}$. Define

$$\hat{\mathscr{K}} = \mathscr{K}_{\mathbb{E}} \setminus \bigcup_{e\in E_0} \bigcup_{r\in Q^+} \mathscr{K}_{e,r}^{n+1},$$

where

$$\mathscr{K}_{e,r}^{n+1} = \{X\in\mathscr{K}_{\mathbb{E}} \mid M_+^{n+1}(X) \cap U_{\mathbb{E}}(e, r) = \varnothing\}.$$

The set $\hat{\mathscr{K}}$ is residual in $\mathscr{K}_{\mathbb{E}}$. In fact, if $A_0\in\mathscr{K}_{\mathbb{E}}$ and $\lambda > 0$ are arbitrary, by virtue of Propositions 1 and 2 (with $e_0 = e$) there exist $B \in \mathscr{K}_{\mathbb{E}}$ and $\sigma > 0$, with $U_{\mathscr{K}_{\mathbb{E}}}(B,\sigma) \subset U_{\mathscr{K}_{\mathbb{E}}}(A_0,\lambda)$, such that for every $X \in U_{\mathscr{K}_{\mathbb{E}}}(B,\sigma)$ we have $M_+^{n+1}(X) \cap U_{\mathbb{E}}(e,r) \neq \emptyset$. Thus $\mathscr{K}_{e,r}^{n+1}$ is nowhere dense in $\mathscr{K}_{\mathbb{E}}$, and hence $\hat{\mathscr{K}}$ is residual in $\mathscr{K}_{\mathbb{E}}$.

Let $X \in \hat{\mathscr{K}}$ be arbitrary. Let $u \in \mathbb{E}$ and $s > 0$ be arbitrary. Take $e \in E_0$ and $r > 0$ so that $U_{\mathbb{E}}(e,r) \subset U_{\mathbb{E}}(u,s)$. Since $X \notin \mathscr{K}_{e,r}^{n+1}$ the set $M_+^{n+1}(X) \cap U_{\mathbb{E}}(e,r)$ is nonempty and, *a fortiori*, $M_+^{n+1}(X) \cap U_{\mathbb{E}}(u,s)$ is so. As $u \in \mathbb{E}$ and $s > 0$ are arbitrary, it follows that $M_+^{n+1}(X)$ is dense in $\mathbb{E}$. This completes the proof.

Theorem 2. *Let $\mathbb{E}$ be a real Hilbert space with* $\dim \mathbb{E} = n \geqslant 2$. *Then, for a typical* $X \in \mathscr{K}_{\mathbb{E}}$,

(i) *the $(n+1)$-valued locus of π_X is dense in $\mathbb{E}$;*
(ii) *the $(n+2)^+$-valued locus of π_X is empty.*

Proof. For $r, p \in \mathbb{N}$ denote by $\mathscr{N}_{r,p}$ the set of all $X \in \mathscr{K}_{\mathbb{E}}$ satisfying the following two properties:

(q_1) there exists $e' \in U_{\mathbb{E}}(\mathbf{0}, r)$ with card $\pi_X(e') \geqslant n+2$;
(q_2) there exists a set $\{x_1, \dots, x_{n+2}\}$ of $n+2$ points $x_r \in \pi_X(e')$ such that $m(\{x_1, \dots, x_{n+2}\}) > 1/p$.

The set $\mathscr{N}_{r,p}$ is nowhere dense in $\mathscr{K}_{\mathbb{E}}$. To see this, given $A_0 \in \mathscr{K}_{\mathbb{E}}$ and $\lambda > 0$ arbitrary, it suffices to show that there exist $B \in \mathscr{K}_{\mathbb{E}}$ and $\sigma > 0$ such that

$$U_{\mathscr{K}_{\mathbb{E}}}(B,\sigma) \subset U_{\mathscr{K}_{\mathbb{E}}}(A_0,\lambda) \cap (\mathscr{K}_{\mathbb{E}} \setminus \mathscr{N}_{r,p}). \tag{3·1}$$

Since A_0 is compact, there is an $A \in \mathscr{P}_{n+k}$, for some $k \in \mathbb{N}$, verifying $h(A, A_0) < \lambda/4$. By Proposition 3 (with $\epsilon = \lambda/4$) there exists $B \in \mathscr{P}_{n+k}$, with $h(B, A) < \lambda/4$, such that each set $\{b_{i_1}, \dots, b_{i_{n+1}}\}$ of $n+1$ different points $b_{i_r} \in B$ satisfies properties (i) and (ii) of Proposition 3. Hence, for every $u \in \mathbb{E}$, there is a set of q points $b_{i_k} \in B$, with $1 \leqslant q \leqslant n+1$, such that

$$\pi_B(u) = \{b_{i_1}, \dots, b_{i_q}\}. \tag{3·2}$$

The mapping $(u, X) \mapsto \pi_X(u)$ from $\mathbb{E} \times \mathscr{K}_{\mathbb{E}}$ to $\mathscr{K}_{\mathbb{E}}$ is upper semicontinuous, hence for each $u \in \hat{U}_{\mathbb{E}}(\mathbf{0}, r)$ there is a $\delta(u) > 0$ such that, for every $v \in U_{\mathbb{E}}(u, \delta(u))$ and $X \in U_{\mathscr{K}_{\mathbb{E}}}(B, \delta(u))$, we have

$$\pi_X(v) \subset \pi_B(u) + U_{\mathbb{E}}\left(\mathbf{0}, \frac{1}{4p}\right). \tag{3·3}$$

As $\tilde{U}_{\mathbb{E}}(\mathbf{0}, r)$ is compact, there is a finite number of points $u_j \in \tilde{U}_{\mathbb{E}}(\mathbf{0}, r)$, $j = 1, \dots, d$, such that

$$\tilde{U}_{\mathbb{E}}(e, r) \subset \bigcup_{j=1}^{d} U_{\mathbb{E}}(u_j, \delta(u_j)). \tag{3·4}$$

Now fix σ so that

$$0 < \sigma < \min\left\{\delta(u_1), \dots, \delta(u_d), \frac{\lambda}{2}\right\}.$$

We will show that, with the above choice of B and σ, (3·1) holds true.

It is evident that $U_{\mathscr{K}_{\mathbb{E}}}(B,\sigma) \subset U_{\mathbb{E}}(A_0, \lambda)$, for $h(B, A_0) < \lambda/2$ and $\sigma < \lambda/2$. It remains to verify that

$$U_{\mathscr{K}_{\mathbb{E}}}(B,\sigma) \subset \mathscr{K}_{\mathbb{E}} \setminus \mathscr{N}_{r,p}. \tag{3·5}$$

To this end, let $X \in U_{\mathscr{K}_{\mathbb{E}}}(B,\sigma)$. If card $\pi_X(e') \leqslant n+1$ for every $e' \in U_{\mathbb{E}}(e, r)$, then (q_1) fails and hence $X \notin \mathscr{N}_{r,p}$. Now suppose that there is an $e' \in U_{\mathbb{E}}(e, r)$ for which card $\pi_X(e') \geqslant n+2$. In view of (3·4), let $e' \in U_{\mathbb{E}}(u_j, \delta(u_j))$, for some j, $1 \leqslant j \leqslant d$. Clearly $X \in U_{\mathscr{K}_{\mathbb{E}}}(B, \delta(u_j))$, for $\sigma < \delta(u_j)$. By virtue of (3·2) and (3·3) (with $u = u_j$), there exists a set of q points $b_{i_k} \in B$, with $1 \leqslant q \leqslant n+1$, such that

$$\pi_X(e') \subset \bigcup_{k=1}^{q} U_{\mathbb{E}}\left(b_{i_k}, \frac{1}{4p}\right).$$

Let $\{x_1, \ldots, x_{n+2}\}$ be an arbitrary set of $n+2$ points $x_i \in \pi_X(e')$. Since there are at most $n+1$ balls $U_{\mathbb{E}}(b_{i_k}, 1/4p)$, at least one of them must contain two (or more) x_is, thus $m(\{x_1, \ldots, x_{n+2}\}) < 1/2p$. Consequently (q_2) fails and again $X \notin \mathscr{N}_{r,p}$, completing the proof of (3·5). Hence $\mathscr{N}_{r,p}$ is nowhere dense in $\mathscr{K}_{\mathbb{E}}$.

Let us prove (ii). We have

$$\bigcap_{r \in \mathbb{N}} \bigcap_{p \in \mathbb{N}} (\mathscr{K}_{\mathbb{E}} \setminus \mathscr{N}_{r,p}) \subset \{X \in \mathscr{K}_{\mathbb{E}} \mid M_+^{n+2}(X) = \varnothing\}. \tag{3·6}$$

In fact, let X be in the set on the left hand side of (3·6). Suppose, on the contrary, that $M_+^{n+2}(X) \neq \varnothing$, and let $e' \in \mathbb{E}$ be such that card $\pi_X(e') \geqslant n+2$. Let $\{x_1, \ldots, x_{n+2}\}$ be a set of $n+2$ points $x_i \in \pi_X(e')$ such that $m(\{x_1, \ldots, x_{n+2}\}) > 0$. Take $r \in \mathbb{N}$ so that $e' \in U_{\mathbb{E}}(\mathbf{0}, r)$. In view of (3·6), for every $p \in \mathbb{N}$ we have $X \notin \mathscr{N}_{r,p}$. Hence

$$m(\{x_1, \ldots, x_{n+2}\}) \leqslant \frac{1}{p},$$

and, as $p \in \mathbb{N}$ is arbitrary, a contradiction follows. Consequently $M_+^{n+2}(X) = \varnothing$, and (3·6) is proved. Furthermore, the set on the right hand side of (3·6) is residual in $\mathscr{K}_{\mathbb{E}}$, for the $\mathscr{N}_{r,p}$s are nowhere dense in $\mathscr{K}_{\mathbb{E}}$. Thus (ii) is true.

The statement (i) is an immediate consequence of Theorem 1 and of statement (ii). This completes the proof.

The following theorem gives a characterization of the dimension of $\mathbb{E}$ by means of a typical property holding in the space of the compact subsets of $\mathbb{E}$.

THEOREM 3. *Let $\mathbb{E}$ be a real separable Hilbert space with* $\dim \mathbb{E} \geqslant 2$. *Then we have*:

(i) $\dim \mathbb{E} = +\infty$ *if and only if for a typical $X \in \mathscr{K}_{\mathbb{E}}$ the $(n+1)^+$-valued locus of π_X is dense in $\mathbb{E}$ for each $n \in \mathbb{N}$;*

(ii) $\dim \mathbb{E} = n$ *if and only if, for a typical $X \in \mathscr{K}_{\mathbb{E}}$, the $(n+1)^+$-valued locus of π_X is dense in $\mathbb{E}$ while the $(n+2)^+$-valued locus of π_X is empty.*

Proof. The statement (i) follows from Theorems 1 and 2. In (ii) the necessity of the condition follows from Theorem 2, since for any typical $X \in \mathscr{K}_{\mathbb{E}}$, $M^{n+1}(X) \subset M_+^{n+1}(X)$. To prove the sufficiency, put $m = \dim \mathbb{E}$ and observe that $m < +\infty$, by Theorem 1. Suppose $m > n$ (if $m < n$ the argument is similar). For any typical $X \in \mathscr{K}_{\mathbb{E}}$ we have $M^{m+1}(X) \subset M_+^{m+1}(X) \subset M_+^{m+2}(X) = \varnothing$. This and Theorem 2(i) (with m in the place of n) yield a contradiction. Hence $m = n$, and also (ii) is proved. This completes the proof.

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