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Abstract. We point out in this paper the common nature of the cut locus studied in differential geometry and the ambiguous locus, a notion belonging to the geometry of Banach spaces. Using a theorem formulated in length spaces, we strengthen an old result of Stechkin in the case of typical compact sets.

Introduction

The notion of cut locus originally appeared in Differential Geometry and was introduced, under the name of "ligne de partage", by H. Poincaré in 1905 [7]. After a quiet period of 30 years, S. B. Myers [5], [6] and then many other authors continued an extensive investigation of the cut locus.

Initially the cut locus was understood as a subset of a Riemannian manifold and was associated to some point of the manifold. After the sporadic consideration of some other objects instead of a point, Shiohama and Tanaka considered the general case of a compact set K to which they associated the cut locus $C(K)$, and gave a detailed description of it in the case of Alexandrov surfaces [10]. These surfaces and their higher dimensional analogon, the Alexandrov spaces with curvature bounded below, simultaneously generalize the Riemannian manifolds and the convex hypersurfaces.

As soon as we leave the Riemannian framework, or as soon as we consider arbitrary compacta instead of points, the cut locus can get such nasty properties like, for example, being dense.

A *length space* X is a metric space in which every pair of points is joined by some arc of length equal to the distance between the points. Such an arc is called a *segment.* The *cut locus* $C(K)$ of the compact set K in a length space X is the set of all points $x \in X$ such that some *segment from x to K*, i.e. a segment whose length minimizes the distance from x to points in K, is included in no segment from some other point to K.

A related set, the *ambiguous locus* $A(K)$ associated to the compact set $K \subset X$ is defined to be the set of all $x \in X$ at which the nearest point mapping from x to K is not single-valued. Part of the analytical investigations of the nearest point mapping, the study of this set in Banach or Hilbert spaces received much attention in recent years. In any Alexandrov space with curvature bounded below and in any strictly convex Banach space $A(K) \subset C(K)$, and we may of course consider closed sets K instead of compact ones.

The possible density of the cut or ambiguous locus was repeatedly discovered, on various occasions. In 1982 we showed that the cut locus of any point on *most* convex surfaces (of arbitrary finite dimension), i.e. on all but a family of first Baire category, is even residual (i.e. contains most points) on those surfaces [13]. In 1987, A. Rivière studied in his Thesis $C(K)$ with K a closed set in a Euclidean space $\mathbb{R}^d$ and provided an accurate description; he also showed that $C(K)$ may be dense in the complement of K (see [9]).

In 1990 we proved that $A(K)$ is dense in $\mathbb{R}^d$ for most compact sets $K \subset \mathbb{R}^d$ [14]. This result was generalized to Banach spaces and succesively refined by De Blasi and Myjak [1], De Blasi and Zamfirescu [2], Zhivkov [16], [17] etc.

In this paper we fill a certain gap. Because all authors except Rivière concentrated on K and not on its complement, where $C(K)$ actually lives, the case of those K with typical convex complement of its interior was not yet investigated.

Ambiguous loci in Minkowski spaces

Let X be a finite-dimensional real smooth and strictly convex Banach space. It is well-known that the space $\mathcal{C}$ of all compact convex sets in X endowed with the Pompeiu-Hausdorff metric is complete.

Theorem 1. *For most compact convex sets* $K \subset X$, *the ambiguous locus* $A(X \setminus \text{int}K)$ *is dense in* K.

Proof. Let $b \in X$ and $r_0 > 0$. We shall prove that the set of all $K \in \mathcal{C}$, for which $B(b, r_0) \subset K$ and the nearest point mapping is single-valued at all points of $B(b, r_0)$ is nowhere dense.

Let $\mathcal{O} \subset \mathcal{C}$ be open. Suppose $K \in \mathcal{O}$ and $B(b, r_0) \subset K$. We approximate K by polytopes including K and find a polytope $P \in \mathcal{O}$.

Let $r_1 = \max\{r : B(b,r) \subset P\}$ and let $c_1 \in \text{bd}P$ satisfy $\|b - c_1\| = r_1$. The point c_1 belongs to a facet F of P, because $B(b, r_1)$ is smooth. If bd$P$ has further points closest to b, replace the corresponding facets by parallel ones farther from b, in order to get uniqueness of the closest point c_1.

Let $\varepsilon' > 0$ be such that for each point in $B(b, \varepsilon')$, there is a unique closest point from b in bd$P$.

Consider the half-space H_2^+ containing b and bounded by the hyperplane H_2 which supports $B(b, r_1)$ at $c_2 \neq c_1$. The points c_1, c_2 can be chosen so close that H_2 meets the relative interior of F and $P' = P \cap H_2^+ \in \mathcal{O}$. Now there are precisely two nearest points from b in the boudary of P', namely c_1 and c_2.

Let b_i belong to the line-segment bc_i $(i = 1, 2)$ such that

$$\|b - b_1\| = \|b - b_2\| = \varepsilon''$$

and

$$\varepsilon'' < \min\{r_0, \varepsilon'/4, (\inf_{z \in P^*} \|b - z\| - r_1)/4\},$$

where $P^* = F \cap H_2$. Let H_1 be a hyperplane containing P^* and some interior point of P', and let A be the set of all points al distance less than ε'' from $H_1 \cap \text{bd}P'$.

Notice that the distance from any point of P^* and also from any point of any facet different from F to any point of the broken line $b_1 b b_2$ is larger than $3\varepsilon''$. Consequently, the distance from any point of A to any point of $b_1 b b_2$ is larger than $2\varepsilon''$.

Because the balls are strictly convex, $\|b_1 - c_2\| > \|b_1 - c_1\|$ and $\|b_2 - c_1\| > \|b_2 - c_2\|$.

Now consider a small neighbourhood $\mathcal{N}$ of P' in $\mathcal{O}$. Let $\varepsilon > 0$ be smaller than ε'', $(\|b_1 - c_2\| - \|b_1 - c_1\|)/2$ and $(\|b_2 - c_1\| - \|b_2 - c_2\|)/2$.

For a convex body $K \in \mathcal{N}$ at Pompeiu-Hausdorff distance less than ε from P', bd$K$ meets both $B(c_1, \varepsilon)$ and $B(c_2, \varepsilon)$, and the sets $(\text{bd}K) \cap B(c_1, \varepsilon)$ and $(\text{bd}K) \cap B(c_2, \varepsilon)$ lie in different components of $(\text{bd}K) \setminus A$.

All points of A are farther away from b_1bb_2 than those of $B(c_1,\varepsilon)$ and $B(c_2,\varepsilon)$, b_1 is closer to any point of $B(c_1,\varepsilon)$ than to any point of $B(c_2,\varepsilon)$, and b_2 is closer to any point of $B(c_2,\varepsilon)$ than to any point of $B(c_1,\varepsilon)$. Therefore there must exist a point in b_1bb_2 with two closest points in bd$K$, one on each side of A.

Hence the set of all $K \in \mathcal{C}$, for which $B(b,r) \subset K$ and the nearest point mapping is single-valued at all points of $B(b,r)$, for some point b of rational coordinates and some positive rational number r, is nowhere dense. This implies that the set of all $K \in \mathcal{C}$, for which $B(b,r) \subset K$ and the nearest point mapping is single-valued at all points of $B(b,r)$, for some nondegenerate ball $B(b,r)$, is of first category.

Ambiguous loci in Euclidean spaces

We give here another proof to Theorem 1, based on curvature properties of typical convex surfaces in $\mathbb{R}^d$.

Second proof of Theorem 1 (in $\mathbb{R}^d$). Let $K \in \mathcal{C}$ be typical. Suppose the nearest point mapping p associated to K is single-valued in the open ball $B(a,r) \subset K$. Then p is continuous there. Using the inclusion $B(a, \|a - p(a)\|) \subset K$ or the fact that, by a result of Klee [4] (see also Gruber [3]), bd$K$ is of class C^1, we see that, for r small enough, the angle between $ap(a)$ and $bp(b)$ remains small for all $b \in B(a,r)$ and the restriction $\tilde{p}$ of p to the disc

$$D = \{x \in B(a,r) : \langle x - a, a - p(a)\rangle = 0\}$$

is bi-Lipschitz, hence a homeomorphism. Thus, $\tilde{p}(D)$ is open in bd$K$ and, for each point $x \in \tilde{p}(D)$, $\rho_i^\tau(x) \geq \|x - \tilde{p}^{-1}(x)\| > 0$ for all tangent directions τ, where $\rho_i^\tau(x)$ denotes the lower radius of curvature at x in direction τ. This contradicts several results on the curvature of typical convex surfaces, for example Theorem 2 in [12] which says that $\rho_i^\tau(x) = 0$ at most points $x \in$ bd$K$, in each tangent direction τ.

Multijoined loci in length spaces

An old result of Stechkin [11] says that the nearest point mapping with respect to any compact set in a Banach space is single-valued at most points of the space. We are going to show that, with respect to most compact sets, the nearest point mapping is not only single-valued, but also injective on a residual set.

This means that, for a typical compact set K, if for any point x in Stechkin's residual set we delete the whole line segment $xp(x)$ except for the endpoint x, we are still left with a residual set!

To use a framework as general as possible, let X be a complete length space with nonbifurcating geodesics. Having *nonbifurcating geodesics* means the following: if two segments have a common endpoint and another common point which is not the other endpoint of any of them, then one of the segments includes the other. Any Alexandrov space with curvature bounded below and in particular any Riemannian manifold and any Hilbert space has this property.

If (X, d) is a length space and K a closed set in X, then the *multijoined locus* $M(K)$ of K is defined as the set of all $x \in X$ admitting at least two segments from x to K. Obviously, $A(K) \subset M(K)$.

Theorem 2. *If K is a closed set in a locally compact complete length space (X, d) with nonbifurcating geodesics, and $M(K)$ is dense in X, then $C(K) \setminus M(K)$ is residual in X.*

Proof. Let X_m be the set of those points $x \in X$ which admit a segment to K, included in a segment from some other point to K, the difference between their lengths being $1/m$. We show that X_m is nowhere dense.

Indeed, let O be an open set in X. Let $x \in O$ be joined by at least two segments to K. Suppose there exists a sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in X_m$, converging to x. Let V be a compact neighbourhood of x, containing some ball $B(x, \varepsilon)$. Then, for integers $m_0 > m$ and n_0 such that $1/m_0 < \varepsilon/3$ and $d(x_n, x) < \varepsilon/3$ for each $n \geq n_0$, all points x_n with $n \geq n_0$ belong to $X_{m_0} \cap V$. Every such point x_n belongs to a segment from some point y_n, at distance $1/m_0$ from x_n, to K. Some subsequence of $\{y_n\}_{n=n_0}^{\infty}$ converges, and so does a subsequence of the corresponding subsequence of segments, towards a segment to K of length $d(x, K) + m_0^{-1}$ and containing x, contradicting the nonbifurcation property of X. Here $d(x, K) = \inf_{z \in K} d(x, z)$. Thus, there is an open neighbourhood of x in O whose points are not in X_m, and so X_m is nowhere dense.

Hence most points of X belong to $C(K)$.

Let now M_n be the set of those points in $M(K)$ from which there are two segments to K at Pompeiu-Hausdorff distance at least $1/n$. We show that $M(K)$ is of first category.

Indeed, let again $O \subset X$ be open. Take $x, y \in O$ such that y belongs to a segment from x to K. If y is the limit of some sequence of points in M_n, then the two sequences of segments from those points to K will produce two distinct segments from y to K (at Pompeiu-Hausdorff distance at least $1/n$ from each other), which is impossible, again because X has nonbifurcating geodesics. Hence a whole neighbourhood of y is disjoint from M_n. Thus M_n is nowhere dense and $M(K)$ of first Baire category.

The proof is finished.

Consequences

This simple theorem together with some other known results has striking consequences.

Let $\mathcal{K}$ be the space of all compact sets in $\mathbb{R}^d$ and $p_K : K \to \mathbb{R}$ the nearest point mapping associated to $K \in \mathcal{K}$.

Lemma 1 [15]. *For most $K \in \mathcal{K}$, at most points $x \in K$, for any open set $O \ni x$ the set of directions $\{\|y-x\|^{-1}(y-x) : y \in O \cap K \setminus \{x\}\}$ is dense in S^{d-1}.*

Combining Stechkin's mentioned result with Theorem 2 in [14], with the present Theorem 2 and with Lemma 1 we obtain the following.

Theorem 3. *For most compact sets $K \subset \mathbb{R}^d$, the nearest point mapping p_K is single-valued and injective on a residual subset of $\mathbb{R}^d$ and $p_K(\mathbb{R}^d \setminus K)$ is of first category in K.*

Let now $\mathcal{C}_b$ be the space of all *convex bodies*, i.e. compact convex sets with interior points, in $\mathbb{R}^d$ and consider the nearest point mapping $q_K : \text{int}K \to \text{bd}K$ for $K \in \mathcal{C}_b$. Thus, $q_K = p_{\text{bd}K}|_{\text{int}K}$. Obviously, $\mathcal{C}_b$ is a Baire space.
The following lemma is part of Theorem 2 in [12].

Lemma 2. *For most $K \in \mathcal{C}_b$, at most points $x \in \text{bd}K$ the upper curvature in any tangent direction is ∞.*

Combining Stechkin's result with Theorems 1 and 2 and with Lemma 2, we obtain the following result which remarkably parallels Theorem 3.

Theorem 4. *For most convex bodies $K \subset \mathbb{R}^d$, the nearest point mapping q_K is single-valued and injective on a residual subset of $\text{int}K$ and $q_K(\text{int}K)$ is of first category in $\text{bd}K$.*

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