Bull. Math. Soc. Sc. Math. Roumanie
Tome 43(91) No. 3-4, 2000, 279-285

Acute triangulations

by
THEODOR HANGAN, JIN-ICHI ITOH AND TUDOR ZAMFIRESCU

Dedicated to the memory of Professor Gheorghe Vrănceanu

Abstract

We consider here triangulations of (2-dimensional) surfaces with all triangles acute. Just providing such triangulations is sometimes surprisingly difficult. We ask several natural questions, and start to give answers.

Keywords: *Acute angle, geodesic triangle, triangulation.*
Mathematics Subject Classification: Primary 52A15, 53A05.

Motto:
Cătrănit alerg cu gândul printre sute de morminte.
Văd cum gerul şi furtuna şterg renume-adânc săpate...
Am ajuns şi mă cutremur, căci mormîntul e fierbinte
De căldura unei inimi care parcă încă bate.

T.Z.

1 Introduction

Triangulations are a basic geometric tool. Restricting ourselves here to 2-dimensional compact surfaces, triangulations are finite sets of triangles satisfying certain natural conditions: the intersection of any two of the triangles is either empty or consists of a vertex or of an edge. The edges of any triangle must be shortest paths.

Now, we raise the question of existence and minimality (least number of triangles used) of *acute triangulations*, for various types of surfaces. This means that the involved triangles must be acute, i.e. all their angles must be less than $\pi/2$.

For example, let us consider the surfaces of the five Platonic solids. Of course the regular tetrahedron, octahedron and icosahedron have their natural acute triangulations. Also in the case of the dodecahedron we can easily produce an acute triangulation by dividing each face in the standard way into five triangles.

But, except for the tetrahedron and for the octahedron, these triangulations are not minimal. And what about the cube?

If the considered surface is homeomorphic to the sphere and smooth, then every vertex of any acute triangulation must obviously have degree at least 5. This immediately implies that any acute triangulation of such a surface contains at least 20 triangles. Any flat torus admits an acute triangulation made up by 18 triangles. However, in some cases, 14 are enough!

All these things suggest a few questions.

2 Problems

Problem 1 Does there exist a natural number N such that every convex surface admits an acute triangulation consisting of at most N triangles? If yes, find the minimal such N.

This question does not seem to be easy. A solution to this problem restricted to smooth surfaces would also be of interest.

Problem 2 What are the smallest numbers t and t_α such that any flat torus has an acute triangulation with at most t triangles, and any standard torus (obtained by rotating a unit circle of centre c around a coplanar, nonintersecting line at distance α from c in the Euclidean 3-space has an acute triangulation with at most t_α triangles?

Problem 3. Find the analogous numbers in the nontrivial cases of Platonic solids, i.e. for the surface of the cube, of the dodecahedron, and of the icosahedron.

Surprisingly difficult seem to us the following problems.

Problem 4 Find the analogous number in the case of all tetrahedral surfaces (not only the regular).

Problem 5 Find the analogous number in the case of all n-gons, n being an arbitrary fixed natural number.

Here the n-gons are 2-manifolds with boundary, parts of the plane.

3 About the platonic solids

In this paper, where we merely formulate the problems, we shall solve Problem 3 for the case of the cube, and give some partial results on other cases.

Theorem 3.1 *The cube admits several acute triangulations with* 24 *triangles, and no acute triangulation with fewer triangles.*

Proof: Figure 1 shows four nonisomorphic triangulations of the cube with 24 acute triangles.

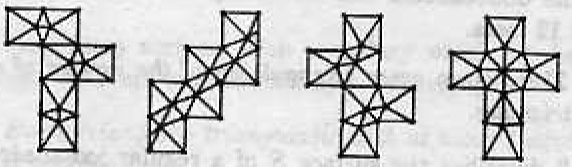

Fig.1

Now let us prove that each acute triangulation of the cube has at least 24 triangles.

Each triangle on the surface S of a cube containing one vertex of the cube in its interior has as sum of its angles $3\pi/2$ and therefore cannot be acute. Since geodesics don't pass through vertices of S, every vertex of the cube must be a vertex of any acute triangulation.

Suppose an acute triangulation has at least 15 vertices. Each vertex which is also a vertex of the cube has degree at least 4, because the total angle around it is $3\pi/2$. Each of the other at least 7 vertices has degree at least 5, the total angle there being 2π. Thus, we have at least $32 + 35$ edges, counted twice. The number of edges, counted twice, equals $3t$, where t is the number of triangles in the triangulation. Hence t is at least 24 (this number must be even).

Suppose now an acute triangulation has at most 14 vertices. Consider the four vertices $\{a, b, c, d\}$ of a face F of the cube. Clearly, some edge must start in a and go through the interior of F. If no vertex of the triangulation is interior to F, then the edge ends outside the interior of F, and either from b or from d no edge can start and go through the interior of F. Hence each face of the cube contains a vertex of the triangulation in its interior. This vertex must then be joined by an edge with each of the four vertices of the face. Indeed, suppose the vertex v interior to some face $abcd$ is not joined with a. Let ae be the edge of the triangulation starting at a and going through the interior of the face $abcd$. If ae meets bc then v must be separated by ae from d in $abcd$. Then either at c or at d appears a nonacute angle and we obtain a contradiction. Let the faces F_1, F_2 have the common edge ab. Let v_1, v_2 be vertices of the triangulation interior to F_1, F_2 respectively. Obviously, if ab is an edge of the triangulation, then v_1v_2 is not, if v_1v_2 is an edge, ab is not, but one of the two edges must be present, otherwise a quadrilateral appears. Hence we may count again the edge, only once, by taking 4 for each vertex of the triangulation interior to some face, plus an edge for every edge of the cube. A priori there might be other edges as well. So, the number of edges is at least $24 + 12$. This means that $3t \geq 72$, whence $t \geq 24$. □

Concerning the dodecaheder we have the following.

Conjecture *There is an acute triangulation of the dodecaheder with less than* 20 *triangles.*

Of course, the dodecaheder can be triangulated by taking as vertices the centers of all its 12 faces.

Theorem 3.2 *There is an acute triangulaton of the surface of a regular icosahedron with 14 triangles.*

Proof: Figure 2 describes the surface S of a regular icosahedron (on the left hand side the upper half of 10 equilateral triangles, on the right hand side the lower half of 10 equilateral triangles).

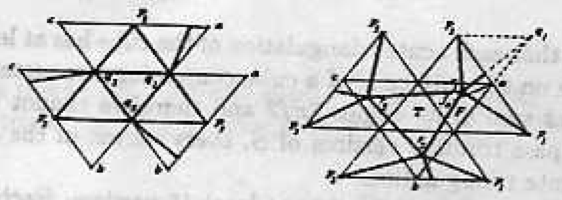

Fig. 2

Consider the face F of the right hand side of Figure 2, which has a vertex at a and a common edge with the central triangle T of the figure. On the line segment from the centre f_a of F to a take a point r_a at small distance ε from f_a. Then, the point r_1 is obtained by rotating r_a around the centre of T counterclockwise. This rotation must be small compared with ε. The points f_b, f_c, r_b, r_c, r_2 and r_3 are obtained analogously. The vertices of our triangulation will be p_1, p_2, p_3 with degree 4 and $q_1, q_2, q_3, r_1, r_2, r_3$ with degree 5. The 21 edges of the triangulation are shown dark in Figure 2.

Let us examine the angles around p_1. The adjacent vertices are q_1, q_2, r_1, r_2. Clearly, $\angle\, q_1p_1q_2 = \frac{\pi}{3}$. Since $\angle\, q_1p_1f_a = \frac{\pi}{2}$, we have $\angle\, g_1p_1r_a < \frac{\pi}{2}$, and so $\angle\, q_1p_1r_1 < \frac{\pi}{2}$ too. Similarly, $\angle\, q_2p_1r_2 < \frac{\pi}{2}$. The angle $r_1p_1r_2$ is acute because is close to $\angle\, f_ap_1f_b = 2\frac{\pi}{6}$.

Next we examine the angles around q_1. The adjacent vertices are p_1, p_3, q_2, q_3, r_1. Of course, $\angle\, p_1q_1q_2 = \angle\, q_2q_1q_3 = \angle\, q_3q_1p_3 = \frac{\pi}{3}$. Also, $\angle\, r_1q_1p_3 < \frac{\pi}{3}$. We have$\angle\, r_1q_1p_1 < \frac{\pi}{2}$ too, because $\angle\, r_1q_1a < \angle\, r_aq_1a < \frac{\pi}{6}$.

Finally, let us examine the angles around r_1. The adjacent vertices are r_2, r_3, p_3, q_1, p_1. Clearly, $\angle\, r_2r_1r_3 = \frac{\pi}{3}$. The angle $p_3r_1r_3$ is acute because it is close to $\angle\, p_3r_ar_c < \angle\, p_3f_af_c = \frac{\pi}{2}$. Similarly, $\angle\, p_1r_1r_2 < \frac{\pi}{2}$. From f_a to q_1 there are two geodesic segments, one of which meets the edge ap_3 in r', say. The angle $p_3r_1q_1$ is close to $\angle\, p_3f_ar' < \angle\, p_3f_aa = \frac{\pi}{3}$, hence acute. Since p_1f_a cuts orthogonally the side of F which contains a but not r', we have $\angle\, p_1f_ar' < \frac{\pi}{2}$. The angle $p_1r_1q_1$ being close to $\angle\, p_1f_ar'$, is also acute.

We investigated all essentially different cases, and thus finished the proof that the considered triangulation is acute. □

4 About polygons

Important and elementary surfaces with boundary which can be considered are planar convex sets. We start with the family of all triangles.

Theorem 4.1 *Every triangle is triangulable with at most 7 acute triangles, and this bound is best possible.*

Proof: Let n be the smallest number of triangles in an acute triangulation of some nonacute triangle. Let abc be a triangle with the angle at a at least $\pi/2$, realizing the above number n of triangles in an acute triangulation. Some edge ad must meet the interior of the triangle. If $d \in bc$, then at least one of the triangles abd and acd is not acute. The triangulation of abc determines a triangulation for the above nonacute triangle, which must have itself at least n triangles. This contradiction shows that d is interior to abc.

If d is joined with both b and c, the obtuse triangle dbc appears, and we get a contradiction as before.

We consider now the case that d is joined with b. We count the edges of the triangulation. The vertex c has degree at least 2, the vertices a and b have degree at least 3, d has degree at least 5, and the other 3 neighbours of d have degree at least 4. So, we obtain together at least 25 edges, counted twice. Hence there are at least 13 edges. Counting the edges from the number n of triangles, those edges which lie on the boundary of the triangle will be counted once, the other edges twice. So, $3n$ plus the number of boundary edges equals at least 26. At most one triangle (that containing c) has two boundary edges. So, there are at most $n+1$ boundary edges. Hence $n + 1 \geq 26 - 3n$, from which it follows that $n > 6$.

The case when d is joined with c is analogous.

Finally we consider the case that d is neither joined with b, nor with c. The vertices b and c have degree at least 2, a has degree at least 3, d has degree at least 5, and the other at least 4 neighbours of d have degree at least 4. Summing up, we get at least 28 edges, counted twice. Counting, as before, the edges from the number of triangles, and observing that at most two triangles can have two boundary edges each, we get the inequality $n + 2 \geq 28 - 3n$, from which follows again $n > 6$.

We construct now an acute triangulation for an arbitrary triangle abc with the angle at a at least $\pi/2$. We consider first the centre i of the circle inscribed to abc, then take the points $c' \in ab$, $a', a'' \in bc$, and $b' \in ca$ such that all the angles $c'ib, a'ib, a''ic, b'ic$ be equal and a little less than $\pi/4$. The line-segments ia, ib', ic', ia', ia'', $c'a'$, and $b'a''$ become edges of the triangulation, which has 7 triangles; we show that it is acute.

Let α, β, γ be the angles of abc at a, b, c respectively.

Clearly, the isosceles triangles $ic'a'$, $ib'a''$, $bc'a'$, $cb'a''$ are acute.

The angle bic equals $\pi - (\beta+\gamma)/2 = (\pi+\alpha)/2$. Thus, the angle $a'ia''$ is slightly larger than $\alpha/2$. Hence it can be arranged to be acute. The angle $c'a'b$ equals

$(\pi-\beta)/2$, so the angle $ia'a''$ is slightly less than $\pi-(\pi-\beta)/2-(\pi/4)=(\beta/2)+(\pi/4)<\pi/2$. The same can be done with the angle $ia''a'$ whence the triangle aib' are acute.

Clearly, $\alpha/2<\pi/2$. The angle $ac'i$ equals the angle $ia'a''$, which we showed to equal $(\beta/2)+(\pi/4)$. The angle aic' equals $\pi-(\alpha+\beta)/2-(\pi/4)=(\gamma/2)+(\pi/4)<\pi/2$. Hence the triangle aic' and, analogously, the triangle aib' is acute.

So, all 7 triangles are acute. □

Theorem 4.2 *Each rectangle is triangulable with 8 acute triangles, and this is the best possible estimate.*

Proof: Let $abcd$ be a rectangle admitting an acute triangulation with n triangles. Obviously, some vertex v of any acute triangulation of $abcd$ must be interior to the rectangle.

If there is no further interior vertex, then there must be some vertices on the edges of the rectangle, because the degree of v is at least 5. Clearly, v is joined with all vertices a,b,c,d. Thus, for every further vertex, there is just one edge going through the interior of the rectangle, whence its degree is 3, and one of the two angles is nonacute. This contradiction shows that there must be further vertices interior to the rectangle. Let p be the number of vertices of the triangulation. Thus, $p\geq 6$. By Euler's formula, the number of edges equals $n+p-1$.

Let i be the number of vertices interior to $abcd$. Then there are precisely $p-i$ boundary vertices, and equally many boundary edges.

Suppose $i=2$. Then there are at least 9 edges interior to the rectangle plus $p-2$ boundary edges. Hence

$$9+p-2\leq n+p-1$$

and $n\geq 8$.

Suppose now $i=3$. Each of these 3 interior vertices has degree at least 5. Summing up, we get 15 edges, among which at most 3 have been counted twice. So there are in fact at least 12 interior edges. And there are exactly $p-3$ boundary edges. Hence

$$12+p-3\leq n+p-1$$

and $n\geq 10$.

Finally, suppose $i\geq 4$. We have at least four vertices of degree at least 5, and four vertices of degree at least 3. So, the number of edges counted twice is at least 32. Since no triangle can have more than one boundary edge, the number of boundary edges is at most n. Hence at most $3n+n$ is the number of edges, counted twice. It follows that $4n\geq 32$, and therefore $n\geq 8$.

Now we present for an arbitrary rectangle $abcd$ an acute triangulation with 8 triangles.

Let p be the midpoint of ab and q the midpoint of cd. We take two points r, s inside $abcd$, close to q and symmetric with respect to pq, such that dr, dq, cq, cs are equally long. The line segments dr, cs, rq, sq, rs, ar, bs, pr and ps determine a triangulation, which is obviously acute. □

Received July 25, 2000

University of Haute-Alsace
Mulhouse
France
E-mail: t.hangan@univ-mulhouse.fr

Faculty of Education
Kumamoto University
Kumamoto 860
Japan
E-mail: j-itoh@gpo.kumamoto-u.ac.jp

Department of Mathematics
University of Dortmund
44221 Dortmund
Germany
E-mail: tudor.zamfirescu@
mathematik.uni-dortmund.de