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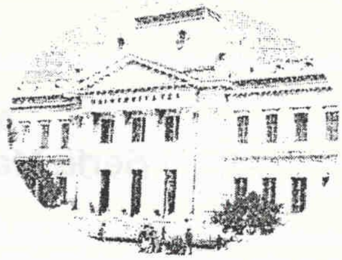

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INTERSECTING LONGEST PATHS OR CYCLES: A SHORT SURVEY

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We talk here only about connected finite graphs without loops and multiple edges. If such a graph is hamiltonian then each longest cycle visits all vertices, so their intersection too contains all vertices. But not all graphs are hamiltonian. For the well-known Petersen graph, each vertex is missed by some longest cycle (in fact it is even hypohamiltonian, which means that the graph itself is not hamiltonian, but the graph minus any of its vertices results in a hamiltonian graph); thus that intersection is empty!

In 1966 Gallai raised the question whether such an example also exists for paths instead of cycles [2]. And indeed, short after, H. Walther found an appropriate example (with 25 vertices) [14]. Hence there is no qualitative difference between the case of cycles and that of paths.

Subsequently the question was refined. Can we ask that any set of k vertices be missed by some longest path (or cycle), for a fixed $k > 1$? Can we ask the graphs to have higher connectivity and still enjoy the above property? Can we impose planarity?

I raised these questions in 1972 [16] and asked explicitly for examples of minimal order. In the following years partial answers appeared. Also, the provided examples were gradually improved.

The P1 problem, i.e. Gallai's original problem about the existence of a graph in which every vertex is missed by some longest path, got a better answer through an example with 12 vertices only (independently found by Walther and Zamfirescu, see [15], [4], [19]). See Figure 1.

The $P1^p$ problem, i.e. the one above restricted to planar graphs, generated several examples, each smaller than the previous one. Walther's first example with 25 vertices was planar, but the smallest so far was found by Schmitz in 1975 and has 17 vertices (see [10]). See Figure 2.

It was conjectured in [17] that 12 and 17 are smallest possible in the problems P1 and $P1^p$ respectively.

The examples are naturally larger if higher connectivity is requested. The first 2-connected example constructed in 1972 for P1 had 82 vertices and was planar [16]. Today, the smallest known 2-connected graphs answering P1 and $P1^p$ have 26 and 32 vertices, respectively [19]. See Figures 3 and 4. The same problem for 3-connected graphs received its first answer in 1974 through Grünbaum's example with 484 vertices [4]. But the best answers so far are in [19] and [5], the respective graphs having 36 and 224 vertices. See the first of them in Figure 5.

The Ck problem, i.e. the existence of graphs in which any k vertices are missed by some longest cycle, was completely solved, in the sense that the provided example has the smallest possible number of vertices (namely $3k + 3$), by Thomassen [11]. See Figure 6. However it can be said that the appropriate frame while working with longest cycles demands connectivity at least 2, and in that case the best known example for the C1 problem remains, as for connectivity 3, Petersen's graph.

The solution to Ck also works for Ck^p. Not so for higher connectivity. For the C1 problem and 2-connected planar graphs, Thomassen found an example with 15 vertices (see [19]). See Figure 7. For 3-connected planar graphs, the first exhibited example belonged to Grünbaum [4] and had 124 vertices. The smallest example known so far is Hatzel's hypohamiltonian planar graph (with 57 vertices) found in 1979 [5]. See Figure 8.

The C2 problem, received – for 2-connected graphs – a positive answer as well. The idea was to insert an "open" copy of a graph G responding to C1 at every vertex of a second graph H of that type. This means to delete one 3-valent vertex of G and replace each vertex of the graph H, which must be cubic, with the rest of G. Thereafter sometimes the edges of H can be contracted. So, a good answer to C2 is the 3-connected graph with 75 vertices obtained by taking both G and H to be Petersen's graph [19]. See Figure 9. Without the contractions mentioned above, the graph has 90 vertices. This was the chronologically first 3-connected example, found in 1974 by Grünbaum [4]. The first 2-connected example had been presented already in 1970 by Walther [14] and had 220 vertices. Concerning planar 2-connected graphs, an example constructed in [17] is the smallest known. It has 135 vertices. See Figure 10. If the graph should even be 3-connected, the first example, with 14818 vertices, appeared in 1976 [19]. The best example was found by Hatzel [5] in 1979 and has 6758 vertices; it uses Hatzel's hypohamiltonian graph and a previously constructed cubic 3-connected graph due to Grünbaum [4], which had answered C1.

The development for the P2 problem was analogous. The most used ideas were to insert an "opened" graph answering C1 in a graph answering P1, or to insert an "opened" graph answering C2 in K_4. So, the first example for P2 was obtained by Grünbaum in 1974 [4]. It was 3-connected and had 324 vertices. Now, the smallest such examples are a 3-connected graph with 270 vertices [19], a 2-connected planar graph with 914 vertices [18], and a 3-connected planar graph with 26378 vertices obtained by Hatzel in 1979 from his corresponding example for C2 [5].

For further historical notes, see [17]. For another rather complete account on these results, see Voss' book [13].

The following interesting questions remained unanswered.

Question 1. Do there exist 4-connected graphs with empty intersection of their longest paths (or cycles)?

Question 2. Do there exist graphs such that any 3 vertices are missed by some longest path (or cycle)?

We may, of course, intersect fewer longest paths than all of them. It is an easy exercise to see that any two longest paths must intersect. Similarly, if 2-connectedness is assumed – and we shall always make this assumption when talking about cycles – any two longest cycles have at least 2 common points. But we already ignore the situation at the next step.

Question 3. Do any 3 longest paths (or cycles) have non-empty intersection?

This being unknown, we introduced numbers p and c meaning the smallest integers such that any p longest paths of any graph intersect, and any c longest cycles of any 2-connected graph intersect. Examples of Schmitz (see [10] and [13]), Jendrol and Skupień [6] show that $2 \leq p \leq 6$ and $2 \leq c \leq 6$, respectively. See Figures 2 and 11.

Question 4. Do any 6 longest paths (or cycles) have non-empty intersection?

We know that two longest paths, and two longest cycles as well, meet. But, in how many points? It depends on the connectivity of the graph. It is immediately seen that k-connectivity implies that there are at least k points in that intersection, for $k = 2$ or 3. Also the case $k = 4$ is easy. The following question was raised by Smith, who conjectured an affirmative answer.

Question 5. Do any two longest cycles in a k-connected graph have at least k common points (for $k \geq 2$)?

Work of Grötschel [3] shows this to be true for $k \leq 6$. However, in general, we only know that there must be $O(\sqrt{k})$ common points [1].

Concerning the C1 problem, we may also ask: Since all most usual families of graphs containing nonhamiltonian members also contain graphs answering C1 in the affirmative, are there any interesting families at all, which have no such graphs?

Work of Menke [7], and Menke, Zamfirescu and Zamfirescu [8] provides and investigates such a family. It contains certain subgraphs of the infinite lattice graph L with vertices at $\mathbb{Z}^2$. More precisely, the family contains graphs of the following type. Take a (finite) cycle C in L. The graph whose vertices and edges are on or inside C is called a *grid graph*. The family of all grid graphs has no example satisfying C1. Concretely, each such graph has at least 4 points lying on each longest cycle. This was established by Menke in 2000 [7].

The family $\mathcal{F}$ of graphs constructed analogously, but starting from the equilaterally triangular lattice, was not yet investigated from this point of view. For a certain subfamily of it, we know that all its graphs but one are hamiltonian [9]. This may be useful in a future investigation of $\mathcal{F}$.

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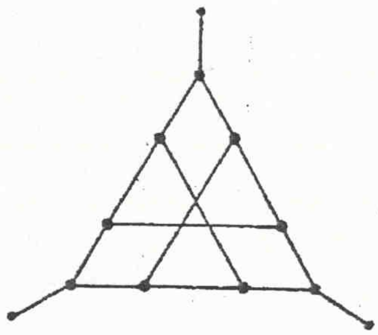

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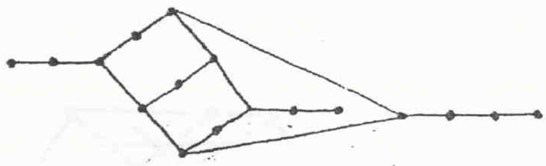

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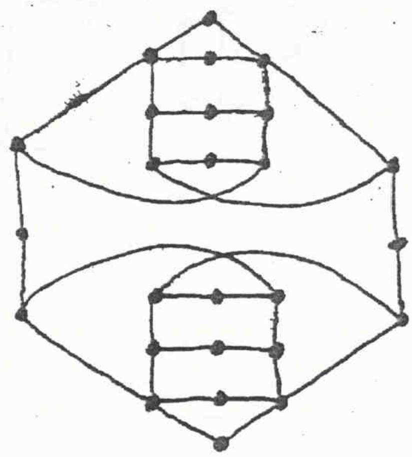

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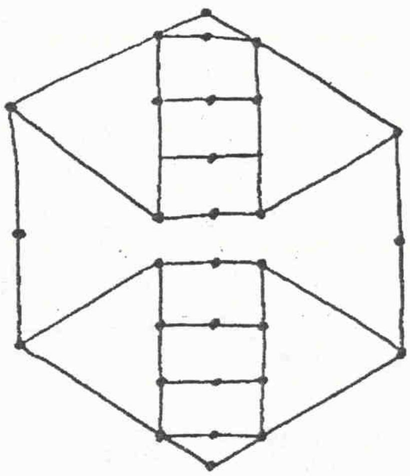

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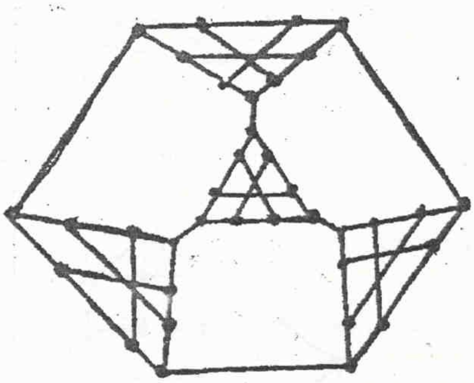

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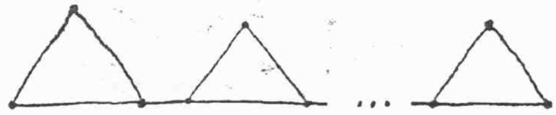

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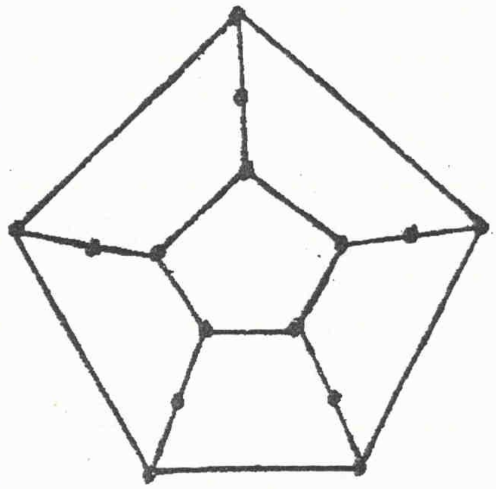

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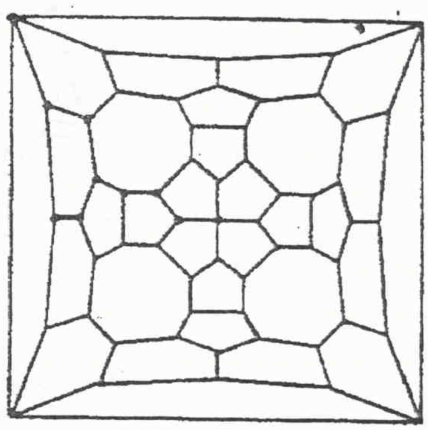

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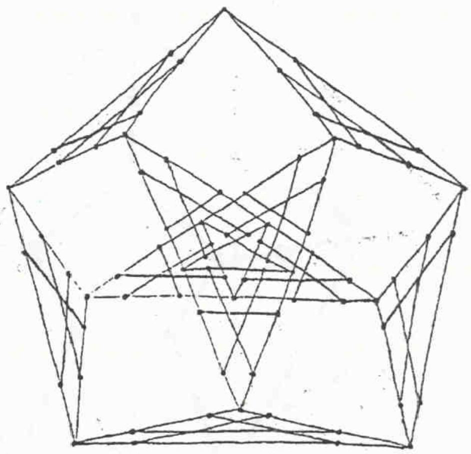

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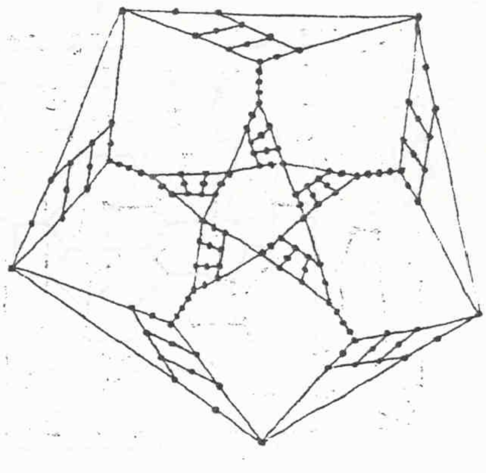

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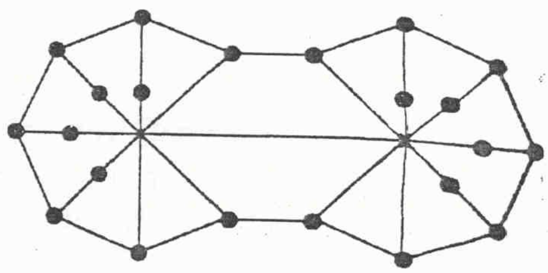

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