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ON THE LENGTH OF THE CUT LOCUS ON SURFACES

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Abstract. Let S be a 2-dimensional torus or projective plane of intrinsic diameter 1. We prove that there always exists a point $p \in S$ the cut locus of which has length at least 2, and make further observations.
MSC 2000: 53C22, 53C20.

Let S be a 2-dimensional compact surface with smooth Riemannian metric, without boundary, of intrinsic diameter 1. Take some point $p \in S$. A point $q \in S$ is called *critical* with respect to the intrinsic distance ρ_p from p if for every tangent direction v at q there is a *segment*, i.e. a shortest path, from p to q whose tangent direction at q makes with v an angle not larger than $\pi/2$.

For each unit tangent vector v at p, we denote the geodesic from p with initial direction v by γ_v. We call the last point q of γ_v up to which the length of the geodesic arc pq equals $\rho_p(q)$ a *cut point* along γ_v. The set of all cut points along geodesics from p is called *cut locus* of p and is denoted by $C(p)$. The topology of S is entirely encoded in the cut locus $C(p)$ (see [5]), and the local structure of $C(p)$ is that of a tree (see [6]). For any point q in $C(p)$ which is not an endpoint (a point of degree 1 in the local tree), there are at least two segments from p to q. We define a subset $C^{cp}(p)$ of $C(p)$ by taking all points $q \in C(p)$ joined with p by at least two segments forming a non-null-homotopic curve, and call it the *cyclic part of the cut locus.* Note that the cyclic part of the cut locus includes the union $C^*(p)$ of all its cycles, and there is a deformation retract from $S \setminus \{p\}$ to $C(p)$ and moreover to $C^{cp}(p)$. It is easily seen that the cyclic part of the cut locus is always connected, while the union of all its cycles may be disconnected. The union of all Jordan arcs joining critical points of ρ_p was called by the first author *essential cut locus* [2], [4]; we denote it by $C^{es}(p)$. Let λA denote the 1-dimensional Hausdorff measure (length) of the set A. It is known that any Jordan arc J between points in $C(p)$ is rectifiable, i.e. λJ is finite (see [3], [6]). Since every cycle contains some critical point, $\lambda C^{cp}(p) \leq \lambda C^{es}(p)$. Put

$$\Lambda(S) = \sup_{p \in S} \lambda C^{es}(p).$$

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On the standard sphere, each cut locus has length 0. However, this is impossible for surfaces of higher genus, or non-orientable. More precisely, we treat here the following problem.

Problem. *Do there exist any positive constants c_1, c_2, depending only on the topology of S, such that*

1) there is a point $p \in S$ with $\lambda C^{cp}(p) > c_1$

2) there is a point $p \in S$ with $\lambda C^{cp}(p) < c_2$?

The two parts of the Problem will receive two different answers.

Theorem 1. *For any compact surface S of diameter 1 not homeomorphic to the sphere, there is a point $p \in S$ such that the length of $C^{cp}(p)$ is at least 1.*

Proof. Let x, y be a pair of points in S at distance 1. The intersection of $C^{cp}(x)$ and $C^{cp}(y)$ is not empty. Indeed, if it were, then the complement of $C^{cp}(x)$ in S, which is homeomorphic to an open disk, would include $C^{cp}(y)$, which could then impossibly have an open disc as complement because it contains a cycle.

Take a point p in the above intersection. Then x belongs to $C^{cp}(p)$, because there are two segments from x to p whose union is not null homotopic. The same is true about y, and there exists an arc in $C^{cp}(p)$ between x and y, of length at least 1.

Corollary. *If the compact surface S has diameter 1 and $\Lambda(S) < 1$, then S is homeomorphic to the sphere.*

We can improve this lower bound in case S is homeomorphic to the projective plane or to the torus. The new bounds are best possible.

Theorem 2. *If the compact surface S of diameter 1 is homeomorphic to the projective plane, then there is a point $p \in S$ such that the length of $C^{cp}(p)$ is at least 2.*

Proof. Let $x, y, p \in S$ be chosen as in the preceding proof; hence x and y belong to $C^{cp}(p)$.

For S homeomorphic to the projective plane, $C^{cp}(q)$ is a single cycle. Then each of the two arcs determined by x, y on $C^{cp}(p)$ has length at least 1, so $\lambda C^{cp}(p) \geq 2$.

In the case of the canonical projective plane of diameter $\pi/2$, every cut locus is a cycle of length π.

Theorem 3. *If the compact surface S of diameter 1 is homeomorphic to the torus, then there is a point $p \in S$ such that the length of $C^{cp}(p)$ is greater than 2.*

Proof. Take x, y, p as before. For S homeomorphic to the torus we have $C^{cp}(q) = C^*(q)$ for all $q \in S$. Hence $x, y \in C^*(p)$. Now, $C^*(p)$ is a union of two cycles meeting at a point z. If x, y belong to the same cycle $C \subset C^{cp}(p)$, then each of the two arcs determined on C has length at least 1, so $\lambda C \geq 2$ and $\lambda C^{cp}(p) > 2$. If x, y do not belong to the same cycle of $C^{cp}(p)$, then let J_1 be a Jordan arc in $C^{cp}(p)$ joining x to z, and J_2, J_3 the two Jordan arcs in $C^{cp}(p)$ joining z to y. Then $J_1 \cup J_2$ and $J_1 \cup J_3$ cannot be both segments because segments do not bifurcate. So, the sum of their lengths is larger than 2.

The following example shows that the bound given by Theorem 3 is best possible.

Example 1. Consider a thin torus embedded in $\mathbb{R}^3$. On the xz-plane take a small circle $(x-1)^2 + z^2 = \varepsilon^2$, and rotate it around the z-axis. When ε tends to 0, the diameter of this torus tends to π and the length of the cyclic part of any cut locus tends to 2π.

From Theorem 3 it follows that $\Lambda(S) > 2$ for all tori S. The following example settles part 2) of the Problem in the negative for orientable surfaces of genus 1.

Example 2. Consider a torus as described below. (See Figure 1.) On the xy-plane take the square $Q = [0,1] \times [0,1]$. Identify $(x, 0)$ with $(x, 1)$ and $(0, y)$ with $(1, y)$ for all $x, y \in [0,1]$ to obtain a flat torus F. The diagonals D_1, D_2 of Q are cycles in F and determine two squares in F. Consider the union L^e of the $2n$ lines $y = i/4n$ $(i = 2, 4, 6, ..., 4n)$, and the union L^o of the $2n$ lines $y = i/4n$ $(i = 1, 3, 5, ..., 4n-1)$. Let U be the set of all points in Q at distance less than $1/4n$ from $D_1 \cup D_2$. Then each of the sets $M^e = Q \cap L^e$ and $M^o = Q \cap L^o \setminus U$ is a closed union of horizontal line segments.

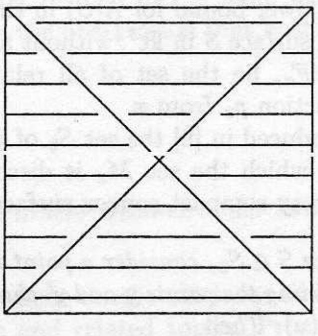

Figure 1.

The continuous function $f : F \to [0,1]$ defined by

$$f(p) = \max\{0, 1 - 4nd(p, M^o)\},$$

where $d(p, M^o)$ denotes the distance from $p \in F$ to M^o, satisfies $f(M^e \cup D_1 \cup D_2) = 0$ and $f(M^o) = 1$. Then the graph of f (with the pertinent identifications), which has diameter less than $(3 + \sqrt{2})/2$, can be approximated by a C^∞ Riemannian surface with the same bound on the diameter and with arbitrarily long $C^{cp}(p)$ for any p, if n is taken large enough.

For surfaces with higher genus, the following example suggests that a larger lower bound for $\Lambda(S)$ might be correct.

Example 3. Let us consider the thin long surface with genus g and diameter 1 obtained by taking the ε-neighbourhood of the 1-dimensional complex shown in Figure 2, with small ε, and approximating it with a Riemannian surface. In this case, for any point p, the length of $C^{cp}(p)$ is about $4 - \frac{2}{g}$ or less.

Figure 2.

We have the folowing conjecture.

Conjecture. *On any oriented surface of genus $g \geq 2$ and diameter* 1*, there is a point p such that $\lambda C^{cp}(p) \geq 4 - \frac{2}{g}$.*

Now we shall present a lower bound for $\Lambda(S)$ in the convex case.

On any compact convex surface S in $\mathbb{R}^3$, without any differentiability assumption, let M_x, respectively F_x, be the set of all relative, respectively absolute, maxima of the distance function ρ_x from x.

The second author introduced in [9] the set $\mathcal{S}_2$ of all compact convex surfaces S admitting a point x for which the set M_x is disconnected. Contrary to the Riemannian case, for arbitrary compact convex surfaces S, $\Lambda(S)$ can be infinite.

Theorem 4. *On the surface $S \in \mathcal{S}_2$, consider a point x with disconnected M_x and the Jordan arc $J \subset C(x)$ joining the points y and y' chosen in different components of M_x. Let $m_J = \inf_{u \in J} \rho_x(u)$. Then*

$$\Lambda(S) \geq \sqrt{\rho_x(y)^2 - m_J^2} + \sqrt{\rho_x(y')^2 - m_J^2}.$$

Proof. Let $z \in J$ satisfy $\rho_x(z) = m_J$. By Lemma 5 in [9], there are precisely two segments from x to z, and these form a closed geodesic arc Γ at x. (This is well-known in the Riemannian case; see [8] for an extension to Alexandrov spaces.)

Let y^*, y'^* be the points of Γ closest to y, respectively y'. Of course, $\rho_x(y^*) \leq \rho_x(z)$. By Alexandrov's Satz 1 in [1], p. 129, $\rho_x(y)$ is smaller than or equal to the length of the hypotenuse of the plane right triangle the other sides of which have lengths $\rho_y(y^*)$ and $\rho_x(y^*)$. Hence

$$\rho_x(y)^2 \leq \rho_y(y^*)^2 + \rho_x(z)^2$$

and

$$\rho_y(y^*) \geq \sqrt{\rho_x(y)^2 - m_J^2}.$$

Analogously,

$$\rho_{y'}(y'^*) \geq \sqrt{\rho_x(y')^2 - m_J^2}.$$

This together with

$$\Lambda(S) \geq \lambda C^{es}(x) \geq \lambda\Gamma \geq \rho_y(y^*) + \rho_{y'}(y'^*)$$

yields the theorem.

C. Vîlcu [7] proved that $\mathcal{S}_2$ is precisely the set of all compact convex surfaces S admitting a point x for which the set F_x is disconnected. Let $r(x)$ be the radius of S in x, i.e. the maximal value of ρ_x.

Theorem 5. *Suppose S is a convex surface in $\mathcal{S}_2$ and Γ is a closed geodesic arc at x separating two components of F_x. Then*

$$\Lambda(S) \geq \sqrt{4r(x)^2 - \lambda\Gamma^2}.$$

Although the arc Γ is not defined in the same way, the proof works like for the preceding theorem, with the observation that both $\rho_x(y^*)$ and $\rho_x(y'^*)$ are not larger than $\lambda\Gamma/2$.

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