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ACUTE TRIANGULATIONS OF TRIANGLES ON THE SPHERE

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Abstract. We prove that each spherical geodesic triangle with angles smaller than π is triangulable with at most 10 acute triangles, and this is the best possible estimate.

MSC 2000: 52A15, 53A05.

The investigation of acute triangulations has one of its origins in a problem of Stover reported by Gardner in 1960 in his Mathematical Games section of the Scientific American (see [4], [5], [6]). There the question was raised, whether a triangle with one obtuse angle can be cut into smaller triangles, all of them acute. Another, even earlier, interest in acute triangulations stems from the discretization of partial differential equations [8].

In 1980, Cassidy and Lord considered acute triangulations for the surface of a square. Maehara recently investigated acute triangulations of quadrilaterals [10] and other polygons, obtaining deeper results.

Acute triangulations with triangles which are close to equilateral were considered by Gerver [6] and, on Riemannian surfaces, by Colin de Verdière and Marin [2].

We are interested only in triangulations all the members of which are *geodesic triangles*, i.e. all edges must be shortest paths. This is motivated by the geometric significance of the *geodesic triangulations*, i.e. those triangulations using geodesic triangles only. Colin de Verdière [1] shows how to change a triangulation of a compact surface of nonpositive curvature into a geodesic triangulation. The planar case was previously treated by Fary [3] and Tutte [12]. From now on *triangulation* will always mean a geodesic one. We focus on triangulations which are *acute*, which means that the angles of all appearing geodesic triangles are smaller than $\pi/2$.

W. Manheimer [11] solved Stover's problem: every non-acute triangle can be triangulated with 7 acute triangles. About fourty years later, H. Maehara [9] showed that each quadrilateral can be triangulated with at most 10 acute triangles.

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Recently, T. Hangan and the authors [7] raised the following problem.

Problem. Does there exist a number N such that every closed convex surface admits a triangulation with at most N acute triangles?

Of course, the same question can be raised for other interesting classes of surfaces, with or without boundary (see [7]). Also, we may well think of higher dimensional generalizations.

Here we consider acute triangulations of spherical triangles. We shall see that the corresponding number is somewhat bigger than in the plane.

Let a, b, c be points on the 2-dimensional Euclidean sphere S^2. In this paper, a *triangle* means an open set O the boundary of which is the union of three shortest paths with all angles (towards O) at most π. If all angles are less than π, the triangle will be called *proper*. Notice that all sides of a proper triangle are less than π.

Let $|xy|$ denote the length of the shortest path $xy \subset S^2$, and $(xy) = xy \setminus \{x, y\}$.

Theorem 1. *Each proper triangle is triangulable with at most 10 acute triangles, and this is the best possible estimate.*

Proof. We start with the following claim.

If there are some points $p \in abc$, $a' \in (bc)$, $b' \in (ca)$, $c' \in (ab)$, such that

$$\angle pa'b = \angle pb'c = \angle pc'a = \pi/2,$$

the angles apb, bpc, cpa are obtuse, and

$$|a'b|, |a'c|, |b'c|, |b'a|, |c'a|, |c'b|, |pa'|, |pb'|, |pc'|$$

are all less than $\pi/2$, then there is a triangulation of abc with 10 acute triangles.

To prove this claim, we make use of Maehara's pivot technique, introduced in [10] for planar triangulations. Consider a small circle of centre p and the circumscribed triangle $a''b''c''$ such that $a''b''$ is orthogonal to pc', $b''c''$ to pa' and $c''a''$ to pb'. If the radius ε of the circle is small enough, then the triangle $a''b''c''$ is acute. We get in this way 10 acute triangles:

$$a''b''c'', ab'a'', ac'a'', bc'b'', ba'b'', ca'c'', cb'c'', a''b'c'', b''c'a'', c''a'b''.$$

Indeed, $\angle pa'c = \pi/2$, $|pa'| < \pi/2$ and $|a'c| < \pi/2$ imply $\angle cpa' < \pi/2$ and $\angle pca' < \pi/2$. Hence $\angle cc''a' < \pi/2$ too, if ε is small enough.

Next, we classify the spherical triangles into the following three types.

(i) At most one edge length is larger than $\pi/2$ and precisely one angle is obtuse or right.

(ii) No angle is obtuse, or all 3 edge lengths are larger than $\pi/2$, or precisely 2 edges have lengths larger than $\pi/2$ and both opposite angles are obtuse or right.

(iii) Precisely two edges have lengths larger than $\pi/2$ and one of the opposite angles is acute.

It is immediately seen that Cases (i), (ii), (iii) cover all possibilities. Now, we provide for each of them an acute triangulation.

In Case (i), we can use the acute triangulation provided for planar triangles in [7].

Let the longest side be ab. Then the angles at a and b are not obtuse. Take the inscribed circle C of centre i. Also, let $a', b' \in ab$, $c'' \in bc$, $c' \in ca$ be such that $|aa'| = |ac'|$, $|bb'| = |bc''|$, and $c'a', b'c''$ are tangent to C. Let $i_a \in bc$, $i_b \in ca$, $i_c \in ab$ be tangency points of C. We get the acute triangles

$$aa'c', bb'c'', ia'c', ib'c'', icc', icc'', ia'b'.$$

Indeed, most involved angles are obviously acute. Less obvious are the following angles: $a'ic'$, cic', $a'ib'$ (and analogous ones).

Since $|ai_b| < \pi/2$ implies $\angle aii_b < \pi/2$, we get $\angle a'ic' = \angle aii_b < \pi/2$.

Next, $|ac'| = |aa'|$ and $|ac| < |ab|$ imply $\angle a^*ii_a < \pi$, where $\{a^*\} = C \cap a'c'$. Thus, $\angle cic' = \angle a^*ii_a/2 < \pi/2$.

Finally, $4\angle a'ib' = 2\pi - \angle i_aii_b$, whence $\angle a'ib' < \pi/2$.

In Case (ii), for all possible situations, we can use the claim at the beginning of the proof, the role of the point p being played by the centre of the circumscribed circle of abc. This can be easily checked.

In Case (iii), let $|ab| > \pi/2$, $|ac| > \pi/2$ and $\angle abc < \pi/2$.

We first observe that in this case $\angle bac < \pi/2$.

Choose $b_3 \in ab$, $a_3 \in b_3c$ and $a', a'' \in aa_3$ such that $|ab_3| = |ac|$, $|a_3b_3| = |a_3c|$, $|aa'| = \pi/2$ and the triangle $a''b_3c$ is equilateral.

Define

$$a_2 = \begin{cases} a'' & \text{if } a'' \in aa' \\ a' & \text{otherwise.} \end{cases}$$

Clearly, a_2b_3c is an acute triangle.

Choose now $b_1 \in ab_3$ and $c_1 \in ac$ with $|ab_1| = |ac_1|$ small. Then choose $b_2 \in ab_3$, $c_2 \in ac$ and $a_1 \in aa_2$, such that $\angle a_2b_2b_3$ and $\angle a_2a_1b_2$ are slightly smaller than $\pi/2$ and $|ab_2| = |ac_2|$. Then all 10 triangles

$$ab_1c_1, a_1b_1c_1, a_1b_1b_2, a_1c_1c_2, a_1a_2b_2, a_1a_2c_2, a_2b_2b_3, a_2c_2c, a_2b_3c, bb_3c$$

are acute.

In order to show that 10 is the best possible estimate, we prove that in one particular case at least 10 triangles are needed.

Assume that all side lengths of the triangle abc are larger than $\pi/2$. We prove that in this case we cannot triangulate with fewer than 10 acute triangles.

Let $\mathcal{T}$ be a triangulation of abc with n points, t triangles and l edges. Each edge of abc must contain a further point: $a' \in bc$, $b' \in ca$, $c' \in ab$. At least two interior edges of $\mathcal{T}$ must start at a', say $a'p, a'q$. At least one interior edge starts at b.

If $p = c'$, then a triangle inside $a'bc'$ has $a'c'$ as a side, and must therefore be obtuse; we got a contradiction. An analogous contradiction is obtained if $p = b'$.

Similarly, if $p = a$, then a triangle inside aba' has aa' as a side and must be obtuse, again a contradiction is obtained.

Hence p and q are distinct interior points.

Analogously, there are two edges starting at each of the points b', c', and ending in the interior of abc.

Suppose p, q are the only interior points obtained as the other endpoints of the interior edges emanating from a', b', c'. As a', b', c' can obviously be joined with some point $r \in S^2$ outside of abc by three arcs having only the point r in common, this would imply that a graph isomorphic to $K_{3,3}$ with vertices a, b, c, p, q, r can be embedded in S^2. This is impossible, by the well-known theorem of Kuratowski, so there must be at least three interior points, whence $n \geq 9$.

Since the degree at a, b, c must be at least 3, the degree at a', b', c' at least 4, and at the interior points at least 5, we have

$$2l \geq 3 \cdot 3 + 3 \cdot 4 + (n - 6) \cdot 5 = 5n - 9.$$

This together with Euler's formula gives

$$2t = 2l - 2n + 2 \geq 3n - 7.$$

Taking into account that $n \geq 9$, we get $t \geq 10$.

There also exist non-proper triangles on S^2. For the reader's convenience we shall treat here one non-proper case, but leave him the pleasure to do the rest.

Theorem 2. *Any non-proper triangle with all side lengths smaller than π is triangulable with 18 acute triangles, and this estimate is best possible.*

Proof. Such a triangle abc must be a hemisphere. Let p be its centre and C a small circle about p. Consider the midpoint a' of bc and the other two analogous points. The edges pa, pa', and the other 4 analogous ones meet C in 6 points. The great circles tangent there to C determine a hexagon. Its sides and the corresponding midpoints of the side of abc determine 6 narrow triangles. Between $a'b$ and two of the preceding narrow triangles there is a further triangle, and there are another 5 of this kind. Finally, p and the sides of the hexagon determine the remaining 6 triangles of the triangulation. It is easily checked that all triangles are acute.

We prove now that no acute triangulation of abc has less than 18 triangles if all side lengths are at least $\pi/2$. First, observe that no triangle of the triangulation with ab as a side would be acute, so there must be some vertex in (ab), and similarly in (bc) and in (ca). All these 6 points must have degree 4. One edge ending in a and one ending in the neighbouring vertex c' of (ab) have a common endpoint a'' inside of pac'. Since there are at least 6 disjoint triangles like pac', we have at least 6 vertices like a'' (including a'').

Suppose there are at least 7 of (each of) them. Then, with the already used notation, at least 7 points have degree at least 4 and the remaining $n-7$ degree at least 5, whence

$$2l \geq 7 \cdot 4 + (n-7) \cdot 5 = 5n - 7.$$

By Euler's formula,

$$2t = 2l - 2n + 2 \geq 3n - 5.$$

Since $n \geq 14$, this implies $t \geq 19$.

Suppose now that there are precisely 6 points like a'', and no further points inside abc. Of course, there are also another 6 points, namely a, b, c and the three on the sides. These form a plane 12-cycle Γ. Since a'' and the other 5 analogous points have degree at least 5, there should be at least 3 diagonals starting at every second point of Γ and coexisting without crossing in the interior of Γ.

The cycle Γ and all edges enclosed by Γ form a planar graph with 12 points, $l' \geq 12 + 9$ edges, and t' triangular faces enclosed by Γ. Clearly, $3t' = 2l' - 12$, whence

$$36 - 3l' + 3t' = 36 - l' - 12 \leq 3.$$

By Euler's formula, we must have here equality, which means that the edges of the graph are precisely the edges of Γ plus 9 diagonals joining all pairs of points analogous to a'' including a'' itself. Delete now from this graph the points of degree 2 (i.e. a, b, c and the three points on the sides of abc). We obtain another planar graph with 6 points only, which is a triangulation of the hexagon, with degree 3 at every vertex of the hexagon and without further points. But this is obviously impossible.

Hence $n \geq 13$. Now,

$$2l \geq 6 \cdot 4 + (n-6) \cdot 5 = 5n - 6,$$

whence

$$2t = 2l - 2n + 2 \geq 3n - 4 \geq 35$$

and $t \geq 18$.

Remark. On surfaces of constant negative curvature, any geodesic triangle admits an acute triangulation with at most 7 triangles.

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