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Acute triangulations: a short survey

Tudor Zamfirescu

1 Introduction

This survey might be of interest also for those members of the Romanian Society of Mathematical Sciences who are teaching in high schools, or simply love elementary geometry. This is so, because several intriguing questions we shall be dealing with here are of that nature.

A *triangulation* of a 2-dimensional space means a collection of (full) triangles covering the space, such that the intersection of any two triangles is either empty or consists of a vertex or of an edge. A triangle is called *geodesic* if all its edges are *segments*, i.e., shortest paths between the corresponding vertices. We are interested only in *geodesic triangulations*, all the members of which are, by definition, geodesic triangles.

Colin de Verdière [6] showed how to transform a triangulation of a compact surface of nonpositive curvature into a geodesic triangulation. The planar case was previously treated by Wagner [25] (see also Fary [8] and Tutte [24]). From now on, *triangulation* will always mean a geodesic one.

Our interest will be focused on triangulations which are *non-obtuse* or *acute*, which means that the angles of all appearing geodesic triangles are not larger than, respectively smaller than, $\pi/2$.

Historical notes

In 1953, R. H. MacNeal showed interest in non-obtuse triangulations as they appeared in connection with the discretization of partial differential equations [19].

The discussion of acute triangulations has one of its origins in a problem of Stover reported in 1960 by Gardner in his Mathematical Games section of the Scientific American (see [9], [10]). There the question was raised whether a triangle with one obtuse angle can be cut into smaller triangles, all of them acute.

Motivated by the proof of the discrete maximum principle, in 1973, Ciarlet, Raviart [5], Strang and Fix [23], and later Santos [22], were also led to non-obtuse triangulations.

Acute triangulations with triangles which are close to equilateral were considered, on Riemannian surfaces, by Colin de Verdière and Marin [7].

Also, Baker, Grosser and Rafferty [1], as well as Bern, Mitchell and Ruppert [2], investigated non-obtuse triangulations of polygons.

Extensions to three dimensions were considered by Křížek and Qun [16], Korotov and Křížek [17], Korotov, Křížek and Neittaanmäki [18].

We shall mainly concentrate here on acute triangulations. In general, non-ontuse triangulations are easier to provide, the constructions are simpler. For a given surface, or class of surfaces, the main problems concern

1) the existence of acute (or non-obtuse) triangulations;

2) the (smallest) number of triangles needed.

Polygons

One direction of research concerns polygons (in the plane). These "flat" surfaces have boundary. Thus, vertices of the triangulation are allowed anywhere on the boundary, not only at the vertices of the polygon. In this direction, Burago and Zalgaller [3] obtained the first results – and very early, in 1960. They proved the existence of acute triangulations, even for complexes of polygons. In [3], they accidentally also answered Stover's problem mentioned above: they showed that any non-acute triangle can be triangulated with 7 acute triangles. (This number is smallest possible.)

In 1980, Cassidy and Lord [4] considered acute triangulations of the square. They showed that it admits a triangulation with 8 acute triangles. In fact, this is true for any rectangle, and the number is best possible [11].

Recently, Maehara investigated acute triangulations of quadrilaterals [20] and other polygons [21]. He proved that any quadrilateral can be triangulated with 10 acute triangles, and also that this bound cannot be improved. However, the exact bound for convex quadrilaterals is still unknown, since Maehara's example requiring 10 acute triangles is non-convex.

Concerning polygons with n sides, Maehara proved that there always exists a triangulation with $O(n)$ acute triangles.

The only "non-flat" surfaces with boundary which have been considered so far are the spherical triangles. Itoh and Zamfirescu [15] established that every such triangle (with angles less than π) can be triangulated with at most 10 acute triangles, and this is best possible.

Regular polyhedral surfaces

Much attention received also the acute triangulation of closed surfaces without boundary.

Suppose the surface S is diffeomeorphic to the sphere S^2. If it admits an acute triangulation $\mathcal{T}$, then the degree of each vertex of $\mathcal{T}$ must be at least 5. The smallest planar graph with minimal degree at least 5 is (the 1-skeleton of) the regular icosahedron. But there is no guarantee that S admits indeed this triangulation with 20 triangles! The sphere S^2, more generally any ellipsoid of revolution, and moreover the nondifferentiable surfaces of the regular icosahedron and dodecahedron (for the latter take its dual), they all do. But many others don't.

We first consider the following problem from [11].

Problem 1. Find the minimal number of triangles of a non-obtuse, respectively acute, triangulation of the Platonic surfaces in the nontrivial cases, i.e., for the surface of the cube, of the regular dodecahedron, and of the regular icosahedron.

In [11] Hangan, Itoh and Zamfirescu proved that the surface of a cube admits several acute triangulations with 24 triangles, and no acute triangulation with fewer triangles.

For the regular tetrahedron and octahedron, their natural triangulation is optimal in the sense that it contains the smallest number of triangles. In [13], Itoh and Zamfirescu treated the case of the regular dodecahedron. They found an acute triangulation with only 14 triangles and proved that there is no such triangulation with less than 12 triangles. The question whether a triangulation with 12 acute triangles does or does not exist is still open.

The case of the regular icosahedron was first considered by Itoh [12], who provided acute triangulations with n triangles for all even numbers $n \geq 16$. In [11], Hangan, Itoh and Zamfirescu described an acute triangulation with 14 triangles only. Finally, in [14], Itoh and Zamfirescu presented a triangulation with 12 acute triangles and proved the impossibility of any smaller one.

Concerning the non-obtuse triangulations, (the surfaces of) the regular tetrahedron and octahedron admit nothing better than their own 1-skeleton; the cube admits a triangulation with 4 triangles, all angles of which are right [14]; the dodecahedron admits a triangulation with 5 acute and 5 right triangles [13]; the icosahedron can be triangulated with 2 acute and 6 right triangles [14]. All these are best possible (regarding the number of non-obtuse triangles).

2 Other closed surfaces

Regarding the arbitrary closed convex polyhedral surfaces, we only know the existence, proved by Burago and Zalgaller [3], of an acute triangulation.

Just in the special case of the doubly covered convex n-gon we have a bound on the needed number of triangles, derived from work of Bern, Mitchell and Ruppert [2] and Maehara [21].

Not even for the family of all tetrahedral surfaces is any number N known, such that each surface in the family admits a triangulation with at most N acute triangles.

And more difficult is, of course, an answer to the following problem, first raised by Hangan, Itoh and Zamfirescu in [11].

Problem 2. Does there exist a number N such that every closed convex surface in $\mathbb{R}^3$ admits an acute triangulation with at most N triangles?

Note that even the existence of an acute triangulation for an arbitrary closed convex surface is not proven so far.

Concerning differentiable surfaces, very little is known. Besides the mentioned case of the ellipsoids of revolution, the only important achievement is the existence result in the mentioned paper by Colin de Verdière and Marin [7].

However, work is in progress. And there are many interesting classes of surfaces, restricted to which Problem 2 is both abordable and challanging.

3 A generalization

We eventually break the usual frame of a survey, and propose a certain generalization. The attempt of triangulating more general metric spaces is tentalizing, but rarely successful. The problems concern the triangle sides, which should be something like segments, and the triangle interiors, whose points need an assignment rule. If we don't care about such things, we can think of the following embedding problem as a generalization of the original triangulation problem.

Let (X, ρ) be a metric space. For any three points $a, b, c \in X$, we say that the angle abc is acute if $\rho(a,c)^2 < \rho(a,b)^2 + \rho(b,c)^2$. A triple $\{a, b, c\}$ is called *triangle* if $\rho(a,c) < \rho(a,b) + \rho(b,c)$ and the other two analogous inequalities hold. A triangle is *acute* if all its three angles are acute.

Problem 3. Given (X, ρ), which combinatorial triangulations can be acutely embedded in X ? And what is the smallest possible number of acute triangles?

(A *combinatorial triangulation* is a finite set of triangles, i.e., triples, each triple being the vertices of a triangle in some usual triangulation of a closed surface. The embedding is just combinatorial. It is acute if the triples, after performing the embedding, become acute triangles.)

Consider, for example, the plane $\mathbb{R}^2$. The smallest triangulation, K_4, cannot be acutely embedded in $\mathbb{R}^2$. But (the 1-skeleton of) the double pyramid over the pentagon can. Just take a regular pentagon $p_1 ... p_5$, and two points q_1, q_2 close to its centre. Then all triangles $p_i p_{i+1} q_j \quad (i = 1, ..., 5, j = 1, 2; p_6 = p_1)$ are acute. Is 10 the smallest possible number of acute triangles?

Exercise. Show that (the 1-skeleton of) the regular octahedron can be acutely embedded in (X, ρ) if X includes a Jordan closed curve.

Problem 3 can be put for any metric spaces, also for discrete ones. So (X, ρ) can be a graph with its usual integer-valued metric. Or even a triangulation!

For the infinite regular triangular lattice in the plane, I see an acutely embedded triangulation (of the projective plane) with 12 triangles. Can you do better? (Yes!)

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Fachbereich Mathematik
Universität Dortmund
44221 Dortmund
Germany

E-mail address: tudor.zamfirescu@mathematik.uni-dortmund.de