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Total Curvature and Spiralling Shortest Paths*

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Abstract. This paper gives a partial confirmation of a conjecture of Agarwal, Har-Peled, Sharir, and Varadarajan that the total curvature of a shortest path on the boundary of a convex polyhedron in $\mathbb{R}^3$ cannot be arbitrarily large. It is shown here that the conjecture holds for a class of polytopes for which the ratio of the radii of the circumscribed and inscribed ball is bounded. On the other hand, an example is constructed to show that the total curvature of a shortest path on the boundary of a convex polyhedron in $\mathbb{R}^3$ can exceed 2π. Another example shows that the spiralling number of a shortest path on the boundary of a convex polyhedron can be arbitrarily large.

1. Introduction

The *total curvature* of a C^2 path C parameterized by arc length s in $\mathbb{R}^n$, $r(s)$, is defined as $\int_C |r''(s)|\, ds$. Fenchel proved in 1929 for $\mathbb{R}^3$ and Borsuk in 1947 for any $\mathbb{R}^n$ that the total curvature of a closed curve is bounded from below by 2π, with the equality holding

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only for convex simple closed curves in $\mathbb{R}^2$. The *total curvature* of a polygonal path $P = [z_0, z_1, \ldots, z_n]$ is defined as

$$t(P) = \sum_{i=1}^{n-1} (\pi - \angle z_{i-1} z_i z_{i+1}).$$

Let $\mathcal{K}$ be the set of all compact convex polyhedra in $\mathbb{R}^3$. Let $\mathcal{T} = \{t(P)\}$, where P is a shortest path joining two points on the boundary of a polyhedron $K \in \mathcal{K}$. It has been asked in [1] whether the set $\mathcal{T}$ is bounded.

We prove here, in Theorem 1 below, that the conjecture holds for polytopes K for which the ratio R/r is bounded from above: here R and r, respectively, are the radii of the circumscribed and inscribed ball to K.

We define the *spiralling number* $s(P)$ for the path P from a to b on the polytopal surface by considering a variable point $x \in P \backslash \{a, b\}$, writing this point in cylindrical coordinates as $(2\pi\varphi(x), r(x), z(x))$ where the z-axis is the line through a and b, and φ is a continuous function. Set now

$$s(P) = \lim_{x \to a, y \to b} |\varphi(x) - \varphi(y)|.$$

This measures how many times the path spirals around the line through a and b.

The proof method of Theorem 1 would work for all polytopes if the function φ had bounded variation. This, however, will be shown to be false. In Section 4 we construct needle-like polytopes K with shortest path P such that $s(P)$ is arbitrarily large. Even more surprisingly, the example can be modified so that P spirals around the line through a and b 100 times in one direction, then 200 times in the opposite direction, then 1000 times in the first direction, etc.

The total curvature $t(P)$ of a planar path P is bounded sharply by 2π and this bound is the lowest possible. A triangle with one of the angles very close to π and two points on the two sides adjacent to the wide angle but close to the vertices at the acute angles provides a simple example.

In Section 3 we construct an example of a shortest path on the boundary of a convex polyhedron in $\mathbb{R}^3$ for which the 2π bound does not hold. On some level the example resembles the planar example involving a triangle with one of the angles close to π.

2. Bounded Total Curvature of Shortest Paths for R/r Bounded

On the boundary of a polytope consider a shortest path P with non-smooth points $z_0, z_1, \ldots, z_n$. Put $x_i = (z_i - z_{i-1})/\|z_i - z_{i-1}\|$ $(i = 1, \ldots, n)$. Let u_i be the outernormal (of the interior) of the facet of z_{i-1} and z_i, and let ξ_i be the angle between x_i and x_{i+1} $(i = 1, \ldots, n-1)$. Then the total curvature of P is $\sum_{i=1}^{n-1} \xi_i$. This can be easily checked. We remark that $x_i - x_{i+1} = \lambda_i(u_i + u_{i+1})$ with $\lambda_i > 0$.

Lemma 1. *If there is a unit vector v such that $u_i v \geq \eta > 0$ for all i, then $\sum_{i=1}^{n-1} \xi_i < \pi/\eta$.*

Proof. First note that $\xi < (\pi/2)\sin\xi$ if $\xi < \pi/2$. Then

$$\sum_{i=1}^{n-1}\xi_i = 2\sum_{i=1}^{n-1}\frac{\xi_i}{2} < 2\sum_{i=1}^{n-1}\frac{\pi}{2}\sin\frac{\xi_i}{2}$$

$$= \frac{\pi}{2}\sum_{i=1}^{n-1}\|x_i - x_{i+1}\| = \frac{\pi}{2}\sum_{i=1}^{n-1}\|\lambda_i(u_i + u_{i+1})\|.$$

Since $\|u_i + u_{i+1}\| \leq 2$ and $(u_i + u_{i+1})v \geq 2\eta$, we have

$$\|u_i + u_{i+1}\| \leq \frac{(u_i + u_{i+1})v}{\eta}.$$

Hence

$$\sum_{i=1}^{n-1}\xi_i < \frac{\pi}{2}\sum_{i=1}^{n-1}\lambda_i\frac{(u_i + u_{i+1})v}{\eta} = \frac{\pi}{2\eta}\sum_{i=1}^{n-1}(x_i - x_{i+1})v = \frac{\pi}{2\eta}(x_1 - x_n)v \leq \frac{\pi}{\eta}. \qquad \square$$

Denote by B the closed unit ball in $\mathbb{R}^3$.

Theorem 1. *Let a polytope Q satisfy $rB \subset Q \subset B$. Then the total curvature of any shortest path on the boundary of Q is less than* $4\pi^2 r^{-2}$.

Proof. Let P be a shortest path between points a and b on the boundary bd Q of Q. The length of P is less than π. Indeed, any plane through a and b intersects the boundary of B along a circle of length at most 2π, and the boundary of Q along a polygon of even smaller length. So one of the two broken lines into which a and b divide the polygon must have length less than π, and P cannot be longer. For a non-zero vector $v \in \mathbb{R}^3$ define the v-shadow $S(v)$ of Q as

$$S(v) = \text{bd}\, Q \cap \left\{\frac{r}{2}B + \lambda v \mid \lambda \geq 0\right\}.$$

Assume that u is the outernormal at an interior point f of a facet F of Q with $f \in S(v)$. We claim that $uv/\|v\| > r/2$. Indeed, the plane Π of F does not meet the interior of rB. Let γ be the angle between u and v. Of course, $\|f\| < 1$. The distance from f to the line through the origin $\mathbf{0}$ and v is at most $r/2$ because $f \in S(v)$. Further, the distance from $\mathbf{0}$ to Π is at least r. Therefore

$$\frac{uv}{\|v\|} = \cos\gamma > \frac{r}{2},$$

which proves the claim. We are going to define points $v_1, v_2, \ldots, v_k$ on P by recursion. Set $v_1 = a$ and assume that v_i has been defined. Let v_{i+1} be the first point on P, going from v_i to b, which is not in the interior of $S(v_i)$. Write P_i for the part of the path P between v_i and v_{i+1}. Since the length of P_i is at least $\|v_i - v_{i+1}\| \geq r/2$ and the length of P is less than π, there are only $k < 2\pi/r$ paths P_i, and we put $v_{k+1} = b$. Note that

Lemma 1 applies to P_i with $\eta = r/2$ by the above claim. Hence the total curvature of P_i is less than $2\pi/r$. Summing up, the total curvature of P is less than

$$k\frac{2\pi}{r} < \frac{4\pi^2}{r^2}. \qquad \square$$

Now we formulate Theorem 1 differently. Let E be the ellipsoid of largest volume included in the polytope Q. As the total curvature is invariant under angle preserving linear transformation, we may assume that E has half-axes a, b, c with $0 < a \le b \le c = 1$. Call Q *needle-like* if b is small, and *pancake-like* if a is small compared with b. Theorem 1 on the boundedness of total curvature holds for convex bodies that are not pancake-like. Only moderate effort is needed to prove the following: if the conjecture holds for needle-like convex bodies, then it holds for pancake-like ones as well. So it would be enough to prove the conjecture for needle-like convex bodies. This would follow if the total variation of φ, the function defined in the first paragraph, were bounded. However, this is not true: an example showing this is given in Section 4.

3. The Total Curvature $t(P) > 2\pi$: An Example

We construct a convex body Δ with two points on the boundary such that the total curvature of the shortest path joining the points exceeds 2π. Δ is constructed in four steps, with certain unbounded convex bodies U, X, and Y_α described in steps 1–3. The Cartesian coordinates of a point $x \in \mathbb{R}^n$ are denoted by $x^{(1)}, \dots, x^{(n)}$.

For $0 < \epsilon < \sqrt{2}/8\pi$ and $i = 1, 2$, define V_i to be the plane $\{(x^{(1)}, x^{(2)}, x^{(3)}) \in \mathbb{R}^3 \mid x^{(3)} = (-1)^i\epsilon\}$. Let A_i be the parabola in $\{(x^{(1)}, x^{(2)}, x^{(3)}) \in V_i \mid x^{(2)} = (x^{(1)})^2\}$ and let B_i be the parabola in $\{(x^{(1)}, x^{(2)}, x^{(3)}) \in V_i \mid x^{(2)} = 2(x^{(1)})^2\}$. Denote by v_i the common vertex of A_i and B_i, by a_i the focus of A_i, and by b_i the focus of B_i. Thus $a_i = (0, \frac{1}{4}, (-1)^i\epsilon)$, $b_i = (0, \frac{1}{8}, (-1)^i\epsilon)$, and $v_i = (0, 0, (-1)^i\epsilon)$. Denote by R_i the convex region in the plane V_i bounded by A_i. Put

$$L = \{(x^{(1)}, x^{(2)}, x^{(3)}) \in \mathbb{R}^3 \mid (x^{(2)} - \epsilon)^2 + (x^{(3)})^2 = 2\epsilon^2,\ x^{(1)} = 0,\ x^{(2)} \le 0\}$$

and

$$\Gamma = \{(x^{(1)}, x^{(2)}, x^{(3)}) \in \mathbb{R}^3 \mid (x^{(2)} - \epsilon)^2 + (x^{(3)})^2 = 2\epsilon^2,\ -\tfrac{1}{2} \le x^{(1)} \le \tfrac{1}{2},\ x^{(2)} \le 0\}.$$

Thus L is a quarter-circle joining the vertices v_1 and v_2 and Γ is a surface obtained by sliding L along the segment $\{(x^{(1)}, x^{(2)}, x^{(3)}) \in \mathbb{R}^3 \mid -\frac{1}{2} \le x^{(1)} \le \frac{1}{2},\ x^{(2)} = x^{(3)} = 0\}$. Note that $|L| = \pi\epsilon/\sqrt{2} < \frac{1}{8}$. Let

$$U = \operatorname{conv}(A_1 \cup A_2 \cup \Gamma)$$

be the convex hull of the union $A_1 \cup A_2 \cup \Gamma$.

Suppose that G is a line in V_2, crossing A_2, and parallel to the directrix of A_2. Denote by δ_B the distance between G and the directrix of B_2, and by δ_A the distance between G and the directrix of A_2. Note that if a shortest path P in bd U joining G with b_1 passes through v_1 and v_2, then the length $|P|$ of P equals $\delta_B + |L|$.

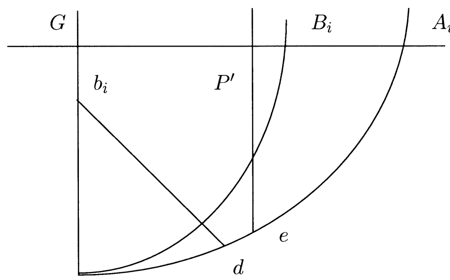

Fig. 1. The parabolas.

Lemma 2. *The shortest path P on the boundary of U joining G with b_1 is unique and passes through the vertices v_1 and v_2.*

Proof. Suppose that P' is a shortest path in bd U joining b_1 with G, different from P, crossing A_1 at $d = (d^{(1)}, d^{(2)}, d^{(3)})$ and A_2 at $e = (e^{(1)}, e^{(2)}, e^{(3)})$, see Fig. 1. Note that $d^{(2)} < e^{(2)}$.

To show that $|P'| > |P|$, consider the two cases: $|e^{(1)}| \geq \frac{1}{2}$ and $|e^{(1)}| < \frac{1}{2}$. We have

1. If $|e^{(1)}| \geq \frac{1}{2}$, then $e^{(2)} \geq \frac{1}{4}$ and $|P'| > \text{dist}(G, e) + \text{dist}(e, a_2) = \delta_A = \delta_B + \frac{1}{8} > \delta_B + |L| = |P|$.
2. If $|e^{(1)}| < \frac{1}{2}$, then P' projects onto P decreasing its length. □

Remark 1. If $P'(e)$ denotes the shortest path in bd U joining b_1 with G and passing through $e \in A_2$, then the length $|P'(e)|$ is a monotone function of $e^{(1)}$ for $0 < e^{(1)} < \frac{1}{2}$ (and for $-\frac{1}{2} < e^{(1)} < 0$).

In the next step of the construction, we modify U to obtain a convex unbounded slab X. Let L' be a convex curve in the $(x^{(1)}, x^{(2)})$-plane close to the segment $(x^{(1)}, 0, 0)$, $-\frac{1}{2} \leq x^{(1)} \leq \frac{1}{2}$, and let Γ' be a positively curved surface in U, close to Γ, obtained by sliding L along L', keeping L parallel to the $(x^{(2)}, x^{(3)})$-plane, and the midpoint of L and L'. Denote by X the convex hull $\text{conv}(A_1 \cup A_2 \cup \Gamma')$. We require of L' (and consequently of Γ') that

1. Γ' contains the quarter-circle L;
2. the path P is the shortest path joining G with b_1 on the boundary of X.

By Remark 1, such an X exists. Note that X is a thin convex unbounded slab whose top and bottom are (horizontal) planar regions R_i''s containing R_i's and with a positively curved side surface close to the vertices v_1 and v_2, see Fig. 2.

For $0 < \alpha < \pi/4$, let Y_α be a "slant slab" obtained from X by "α-slanting" the plane V_1 at the vertex v_1 in the direction perpendicular to the $x^{(1)}$-axis as follows: Let $\Omega_\alpha(x) = (x^{(1)}, x^{(2)} \cos\alpha, -\epsilon - x^{(2)} \sin\alpha)$ and let $Y_\alpha = \text{conv}(X \cup \Omega_\alpha(R_1'))$. Note that the portion of Y_α above the $(x^{(1)}, x^{(2)})$-plane is identical to that of X.

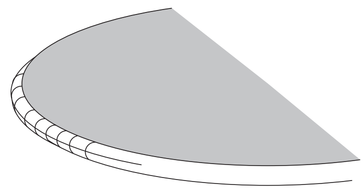

Fig. 2. The unbounded slab X.

Lemma 3. *The shortest path $\tilde{P}$ joining G with $b_1' = \Omega_\alpha(b_1)$ on the boundary of Y_α passes through the vertices v_1 and v_2.*

Proof. A path P' joining G with b_1' in bd Y_α can be mapped (keeping the first coordinate and the distance to the line $(t, 0, -\epsilon)$ unchanged) onto the boundary of X to a path joining G with b_1. Such a projection does not increase the length of the path. Hence, if $P' \neq \tilde{P}$, $|P'| > \delta_B + \pi\epsilon = |P| = |\tilde{P}|$. □

Remark 2. For $\tilde{P}$ as defined in Lemma 3, the total curvature of $\tilde{P}$, $t(\tilde{P})$, equals $\pi - \alpha$.

Denote by S the side boundary of Y_α, i.e., $S = \text{bd}\ Y_\alpha \backslash (R_2' \cup \Omega_\alpha(R_1'))$. For a point w, $\text{conv}(Y_\alpha, w)$ denotes the convex hull of Y_α and w. For a point $w \notin Y_\alpha$, define an *attached cone* with vertex w, $\text{con}(w)$, as the closure of $\text{conv}(Y_\alpha, w) \backslash Y_\alpha$ provided that it does not intersect $R_2' \cup \Omega_\alpha(R_1')$.

Denote by P_0 the shortest path in S, a quarter-circle, joining v_2 with $v_3 = (0, -\epsilon, 0)$. Let w_1, w_2, and w_3 be points close to P_0 for which attached cones $\text{con}(w_i)$ are defined and are pairwise disjoint. Note that $\tilde{Y} = Y_\alpha \cup \text{con}(w_1) \cup \text{con}(w_2) \cup \text{con}(w_3)$ is convex.

First choose points w_1, w_2, and w_3 so that

1. each $\text{con}(w_i)$, $i = 1, 2, 3$, intersects P_0 at one point t_i with $\text{dist}(t_1, t_2) = \text{dist}(t_2, t_3)$;
2. $\text{con}(w_1)$ and $\text{con}(w_3)$ are on the same side P_0, and $\text{con}(w_2)$ is on the other side of P_0.

Then move the points w_1, w_2, and w_3 slightly towards P_0 to obtain points w_1', w_2', and w_3' such that the attached cones $C_i = \text{con}(w_i')$ intersect P_0 and the shortest path K joining G and b_1' in bd $\tilde{Y}$ crosses each C_i, but passes through v_1, v_2, and v_3, see Fig. 3.

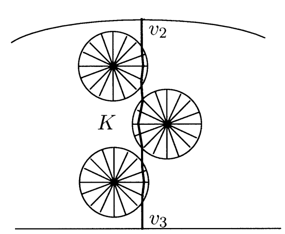

Fig. 3. Attached cones.

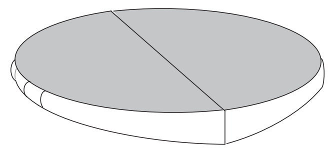

Fig. 4. The double slab Δ.

Note that K is not planar and as follows from the work of Fenchel [3] (alternatively, see [2] or [4]) its total curvature is greater than that of P_0. We have

Lemma 4. *The total curvature* $t(K) = \pi + \beta - \alpha$, *where* $\beta > 0$.

Finally, take $\alpha < \beta$. The line G consists of points (t, g, ϵ), where $-\infty < t < \infty$. Let Δ_1 be the part of $\tilde{Y}$ cut off by the plane $x^{(2)} = g$. Let Δ_2 be a symmetrical copy of Δ_1 and $\Delta = \Delta_1 \cup \Delta_2$ with Δ_1 and Δ_2 glued along the side $x^{(2)} = g$ (see Fig. 4). Let K'' and b_1'' be the path and the point in Δ_2 corresponding to K' and b_1'. For sufficiently large g, the path $\bar{K} = K' \cup K''$ is the shortest path joining b_1' and b_1'' in Δ. We have $T(\bar{K}) = 2(\pi - \beta + \alpha) > 2\pi$. Clearly, Δ is convex.

A polyhedral example can be obtained by a suitable approximation of Δ.

Theorem 2. *There exist a convex polyhedron* $M \subset \mathbb{R}^3$ *with two points* x *and* y *on the boundary of* M *such that the total curvature of the* (*unique*) *shortest path joining* x *and* y *exceeds* 2π.

4. Spiralling Shortest Paths

Here we construct polytopal surfaces possessing shortest paths of arbitrarily large spiralling number. The intrinsic metric on these surfaces will be denoted by δ.

Theorem 3. *Let* n *be an integer. There exist a convex polytope* Q *and a shortest path* P *between two points on the boundary of* Q *with* $s(P) \geq n$.

Proof. Our example is the boundary of the convex hull of a family of equilateral triangles, each two of which have pairwise parallel edges. Suppose the construction performed up to the equilateral triangle abc, so that, from a fixed point x_0 of the (already constructed) surface, the intrinsic distance $\delta(x_0, bc)$ to bc (which is $\min_{y \in bc} \delta(x_0, y)$) is smaller than the distance to any other side of abc. Let ε be the smaller of the two differences. We now construct the next triangle $a'b'c'$. We do so that

$$\delta(x_0, a'b') < \min\{\delta(x_0, b'c'), \delta(x_0, c'a')\}$$

and any shortest path from x_0 to any point w of $a'b'$ necessarily crosses bc. Let z' be such that $\text{conv}\{z', a, b, c\}$ is a regular pyramid containing the already constructed surface.

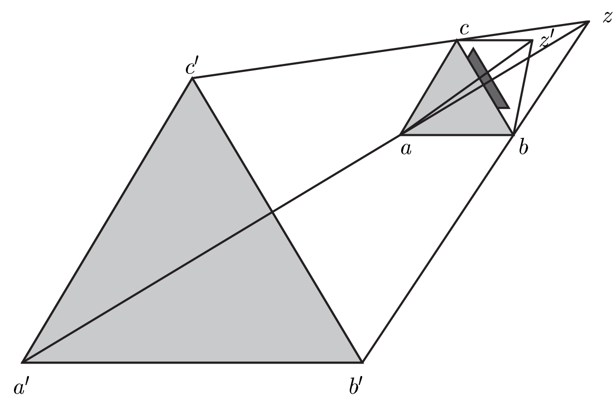

Fig. 5. Stacking triangles.

Then we can find a point z behind z' (see Fig. 5) but close to the pyramid axis such that

$$\|z-a\| = \|z-b\| < \|z-c\|$$

and $\text{conv}\{z, a, b, c\}$ includes $\text{conv}\{z', a, b, c\}$. We shall choose $q > 3$ and put $a' = z + q(a-z)$, $b' = z + q(b-z)$ and $c' = z + q(c-z)$. The angles $\alpha = \angle abb'$ and $\gamma = \angle cbb'$ satisfy

$$\pi/2 < \gamma < \alpha$$

if z is close enough to the pyramid axis.

Consider an unfolding of the surface on the plane of a, b, a', b' without cutting along bb', but cutting along aa', and keep the notation. Let s be the orthogonal projection of c on the line L through a and b. If $\alpha + \gamma - \pi$ is small enough, which is the case if z is far enough, then $\delta(c, s)$ is as small as desired, in particular smaller than $\varepsilon/2$. Let $v \in ab$, $s' \in bs$, see Fig. 6.

It is easily verified that $\delta(s', w) - \delta(v, w)$ increases when w moves from b' to a'. Thus

$$\delta(s', w) - \delta(v, w) \leq \delta(s', a') - \delta(v, a').$$

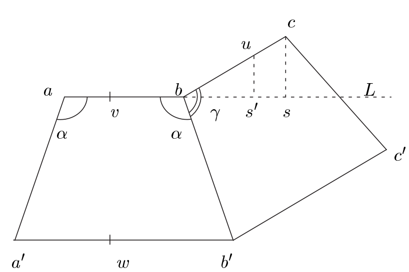

Fig. 6. Unfolding surface.

If $\alpha + \gamma - \pi$ is small enough and q large enough, then

$$\delta(s, a') - \delta(a, a') < \frac{\varepsilon}{2}.$$

Since $\delta(v, a') \geq \delta(a, a')$ and $\delta(s', a') \leq \delta(s, a')$,

$$\delta(s', a') - \delta(v, a') < \frac{\varepsilon}{2}.$$

Hence

$$\delta(s', w) < \delta(v, w) + \frac{\varepsilon}{2}.$$

This ensures that any shortest path from x_0 to some point $w \in a'b'$ crosses bc. Indeed, if the above path crosses ab at v, then

$$\begin{aligned}
\delta(x_0, v) + \delta(v, w) &\geq \delta(x_0, ab) + \delta(v, w) \\
&\geq \delta(x_0, bc) + \varepsilon + \delta(v, w) = \delta(x_0, u) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + \delta(v, w) \\
&> \delta(x_0, u) + \delta(c, s) + \delta(s', w) \geq \delta(x_0, u) + \delta(u, s') + \delta(s', w),
\end{aligned}$$

u being a point of bc closest to x_0, and s' the orthogonal projection of u on L; we got a contradiction. If the path crosses ca at v', say, then

$$\delta(x_0, v') + \delta(v', w) \geq \delta(x_0, ca) + \delta(v', w) \geq \delta(x_0, bc) + \varepsilon + \delta(v, w)$$

and a contradiction is obtained as above.

Now, we have

$$\delta(b, b'c') - \delta(b, a'b') = \delta(b, b')(\sin\gamma - \sin\alpha).$$

By choosing q large enough, the above difference can be made as large as wished; we make it larger than ε. Suppose now a (rectifiable) path from x_0 to $b'c'$ crosses abc at y. Then its length is at least

$$\begin{aligned}
\delta(x_0, y) + \delta(y, b'c') &\geq \delta(x_0, u) + \delta(b, b'c') \geq \delta(x_0, u) + \delta(b, a'b') + \varepsilon \\
&> \delta(x_0, u) + \delta(c, a'b') \geq \delta(x_0, u) + \delta(u, a'b') \geq \delta(x_0, a'b'),
\end{aligned}$$

so it cannot be a shortest path from x_0 to $a'b'c'$. Analogously, no path from x_0 to $c'a'$ is a shortest path from x_0 to $a'b'c'$. This completes the proof of all desired properties for P. It is clear that, iterating this procedure, the shortest path from x_0 to the last constructed triangle has a steadily increasing spiralling number (by a rate close to $\frac{1}{3}$ for each new triangle). Thus $s(P)$ can be made as large as we wish. The theorem is proved. □

Remark 3. In the above construction the path P can be forced to turn, each time, right or left, as wished.

In [5], Pach gives a neat example of a polytope $K \subset \mathbb{R}^3$ and a shortest path P on the boundary of K with non-smooth points $z_0, \ldots, z_n$ such that $\sum_{i=1}^{n} \gamma_i$ is not bounded. Here γ_i is the angle between the outer normals of the facets containing the segments $[z_{i-1}, z_i]$ and $[z_i, z_{i+1}]$. We mention that our construction has the same property.

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