Qualitative infinite version of Erdős' problem about empty polygons

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The well-known problem of Erdős-Szekeres in Discrete Geometry is about realizations of convex polygons P with a given number of vertices in a finite set $F \subset \mathbb{R}^2$, which means that $V(P)$, the vertex set of P, is a subset of F [5], [6]. See the very good survey [11].

This problem admits an obvious generalization to higher dimensions, observed already in [5], and further studied in [3], [12], [10], [14], [8], [9].

Another possible generalization consists in dropping the convexity assumption.

A *polyhedron* is a finite union P of $(d-1)$-dimensional convex polytopes in $\mathbb{R}^d$, such that

1) the intersection of any two of them is either empty or a face of each of them,

2) any point of P has a neighbourhood in P homeomorphic to $\mathbb{R}^{d-1}$,

3) the complement of P in $\mathbb{R}^d$ has precisely two components, one of which is bounded and called *interior* of P.

The facial structure of P determines through its inclusions the combinatorial class of P. The above $(d-1)$-dimensional polytopes are special faces of P called *facets*; if they are simplices then P is *simplicial*.

It is possible to sharpen the higher-dimensional Erdős-Szekeres problem by looking for realizations in a given combinatorial class. Notice that combinatorially distinct polytopes and a fortiori combinatorially distinct polyhedra may have the same number of vertices.

Let $\mathcal{P}$ be the space of all polyhedra and $\mathcal{K}$ the space of all compact sets in $\mathbb{R}^d$. With the Pompeiu-Hausdorff distance ρ, $\mathcal{K}$ is complete. If $K, K' \in \mathcal{K}$ and $\rho(K, K') < \varepsilon$, we say that K' is an *ε-neighbour* of K.

A compact set is called *typical* if it has all typical properties; and a property $\mathbf{P}$ is *typical* if the set of all compact sets not enjoying $\mathbf{P}$ is of first Baire category in $\mathcal{K}$.

We say that a polyhedron P is *realizable* in some set $S \in \mathbb{R}^d$ if there is a polyhedron with vertices in S combinatorially equivalent to P.

More specially, we say that P is *realizable with empty interior* in S if there is a polyhedron with vertices in S combinatorially equivalent to P, the interior of which does not meet S.

A variant of the Erdős-Szekeres problem, due to Erdős, deals with polygons realizable with empty interior in finite subsets of the plane [4]. Generalizations to higher dimensions were studied by Valtr [14], Bisztriczky and Soltan [1], Bisztriczky and Harborth [2].

In the planar case, and similarly in higher dimensions, if n is large enough, there are arbitrarily large finite sets, and of course infinite sets, in which no convex n-gon is realizable with empty interior. Concretely, in the planar case, this is so for $n \geq 7$. And in 3 dimensions, for $n \geq 22$.

Let us remain for a moment in the plane, and keep thinking of the Erdős problem. It seems difficult to pick up empty polygons realized in an arbitrary compact set. But typical compact sets are nowhere dense; this might offer a slight chance...

A polyhedron will be called *combinatorially convex* if it is combinatorially equivalent to the boundary of a convex polytope.

For $M \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$, let

$$\Delta_M(x) = \{\| y - x\|^{-1}(y - x) : y \in M \setminus \{x\}\}.$$

We investigate now the realization of polytopes in a typical compact set.

Two results, one by Wieacker and the other by Gruber, will be especially useful.

Lemma 1 [15]. *A typical compact set contains no affinely dependent sets.*

Lemma 2 [7]. *Let $K \in \mathcal{K}$. A typical compact set contains, for any $\varepsilon > 0$ a subset homothetic to a compact ε-neighbour of K.*

These lemmas immediately imply the following theorem.

Theorem 1. *Every simplicial polyhedron, and no other polyhedron, is realizable in a typical compact set.*

Proof. Let P be a simplicial polyhedron. A small perturbation of P leaves P in the same combinatorial class $\mathcal{C}$, i.e. if $\eta > 0$ is small enough, $P' \in \mathcal{P}$, $\mathrm{card}V(P) = \mathrm{card}V(P')$ and $\rho(P, P') < \eta$, then $P' \in \mathcal{C}$. Let $\varepsilon > 0$. By Lemma 2, a typical compact set contains a set homothetic to a finite set F satisfying $\rho(F, V(P)) < \varepsilon$. For $\eta > 0$, we can choose, for each vertex v of $V(P)$, a point $f(v)$ in F at distance smaller than η from v. For $\eta < \varepsilon$ small enough, f is injective. The set $f(V(P)) \subset F$ is an η-neighbour of $V(P)$, and there is a polyhedron P' with $V(P') = f(V(P))$, itself an η-neighbour of P, and therefore in $\mathcal{C}$ for η small enough. This yields the realization.

The realization of nonsimplicial polyhedra in a typical compact set would contradict Lemma 1.

Corollary 1. *Among the points of a typical compact set we can find, for every $n \geq d+1$, the vertices of a simplicial convex polytope with n vertices.*

This was the pendant of the Erdős-Szekeres situation. Now we turn to the empty-polyhedron problem, pendant of the Erdős problem. Then the requirement is that their interior does not meet the typical compact set.

We obtain here a somewhat surprising result.

Theorem 2. *Every combinatorially convex simplicial polyhedron and no other polyhedron is realizable with empty interior in a typical compact set.*

The proof of Theorem 2 will use the following lemma.

Lemma 3 [16]. *If K is a typical compact set, $x \in K$ and N is a neighbourhood of x, then $\Delta_{K \cap N}(x)$ is dense in some hemisphere of $\mathbb{S}^{d-1}$.*

Proof of Theorem 2. Let the combinatorially convex simplicial polyhedron P and the boundary P^* of a simplicial convex polytope belong to the same combinatorial class $\mathcal{C}$. It is enough to show that the set of all $K \in \mathcal{K}$ in which P is not realizable is nowhere dense.

Indeed, let $\mathcal{O}$ be open in $\mathcal{K}$.

We find a finite set $F \in \mathcal{O}$. Let P' be a small homothetic copy of P^* such that $F \cap P' = \emptyset$ and $F \cup V(P')$ still belongs to $\mathcal{O}$.

Now, for $\varepsilon > 0$ very small, any compact set C in an ε-neighbourhood of $F \cup V(P')$ is still in $\mathcal{O}$ and contains $\mathrm{card}V(P)$ suitable points close to those in $V(P')$, such that the boundary of their convex hull Q belongs to $\mathcal{C}$ and $\mathrm{int}Q$ includes no further points of C. This can be seen in the following way.

First, we choose ε so that each polytope P'' with $\mathrm{card}V(P'') = \mathrm{card}V(P)$ and

$$\rho(V(P''), V(P')) < \varepsilon$$

belongs to $\mathcal{C}$. This implies that, for each $x \in V(P')$, $B(x,\varepsilon) \cap P'(x,\varepsilon) = \emptyset$, where

$$P'(x,\varepsilon) = \mathrm{conv} \bigcup_{y \in V(P') \setminus \{x\}} B(y,\varepsilon).$$

Then, we choose from C, in each ball $B(x,\varepsilon)$ with $x \in V(P')$, a point closest to some hyperplane separating $C \cap B(x,\varepsilon)$ from $P'(x,\varepsilon)$.

Hence P is realizable with empty interior.

Why can no simplicial polyhedron which is not combinatorially convex be realized with empty interior?

Consider a realization of such a polyhedron P. Then, at some vertex v of P, for any neighbourhood N of v, $\Delta_{P \cap N}(v)$ is not contained in any hemisphere of $\mathbb{S}^{d-1}$. This together with Lemma 3 implies that points of K must lie in the interior of P; thus the realization cannot be with empty interior.

The following corollary is a direct strengthening of Corollary 1.

Corollary 2. *Among the points of a typical compact set K we can find, for every $n \geq d+1$, the vertices of a simplicial convex polytope with n vertices, which does not contain any further points of K in its interior.*

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