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Acute Triangulations of the Regular Icosahedral Surface*

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Abstract. We prove here that the surface of the regular icosahedron can be triangulated with 8 non-obtuse and with 12 acute triangles. We also show these numbers to be smallest possible.

1. Introduction

In 1953 MacNeal showed interest in non-obtuse triangulations as they appeared in connection with the discretization of partial differential equations [17].

The discussion of acute triangulations has one of its origins in a problem of Stover reported in 1960 by Gardner in his Mathematical Games section of the *Scientific American* (see [8] and [9]). There the question was raised whether a triangle with one obtuse angle can be cut into smaller triangles, all of them acute. In the same year, independently, Burago and Zalgaller [2] investigated in considerable depth acute triangulations of polygonal complexes, being led to them by the problem of their isometric embedding into $\mathbb{R}^3$. (Accidentally, their paper also includes a solution to Stover's problem!)

Motivated by the proof of the discrete maximum principle, in 1973 Ciarlet and Raviart [4] and Strang and Fix [21], and later Santos [20], were also led to non-obtuse triangulations.

In 1980 Cassidy and Lord [3] considered acute triangulations of the square. Recently, Maehara investigated acute triangulations of quadrilaterals [18] and other polygons [19].

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Acute triangulations with triangles which are close to equilateral were considered by Gerver [10] and, on Riemannian surfaces, by Colin de Verdière and Marin [6]. Also, Baker et al. [1] investigated non-obtuse triangulations of polygons. Extensions to three dimensions were considered by Křížek and Qun [14], Korotov and Křížek [15] and Korotov et al. [16].

A *triangulation* of a two-dimensional space means a collection of (full) triangles covering the space, such that the intersection of any two triangles is either empty or consists of a vertex or of an edge. A triangle is called *geodesic* if all its edges are *segments*, i.e., shortest paths between the corresponding vertices. We are interested only in *geodesic triangulations*, all the members of which are, by definition, geodesic triangles.

Colin de Verdière [5] shows how to transform a triangulation of a compact surface of non-positive curvature into a geodesic triangulation. The planar case was previously treated by Wagner [23] (see also [7] and [22]). From now on, *triangulation* will always mean a geodesic one.

Our interest is focused on triangulations which are *non-obtuse* or *acute*, which means that the angles of all appearing geodesic triangles are not larger than, respectively smaller than, $\pi/2$.

We started together with Hangan in [11] the investigation of acute triangulations of all Platonic surfaces, which are the surfaces of the five well-known Platonic solids.

For the regular tetrahedron and octahedron, their natural triangulation is optimal in the sense that it contains the smallest number of triangles. Among the remaining non-trivial cases only the cube was treated completely [11]. This study was continued for the case of the regular icosahedron by Itoh [12], who provided triangulations with n triangles for all even numbers $n \geq 16$. Also, in [13], we treated the case of the regular dodecahedron, completely settled the non-obtuse case and found a surprisingly small acute triangulation (with 14 triangles only). The question whether a triangulation with 12 acute triangles does or does not exist is still open.

Here we treat the case of the regular icosahedron. We regard our work as a small step towards a solution to the following problem first raised in [11]. We consider this problem very natural, and far from trivial.

Problem 1. Does there exist a number N such that every compact convex surface in $\mathbb{R}^3$ admits an acute triangulation with at most N triangles?

Of course, Problem 1 can be extended (or restricted) to other families of surfaces (such as Riemannian), with or without boundary. Even more generally, families $\mathcal{F}$ of two-dimensional triangulable compact topological spaces may be considered. In particular, $\mathcal{F}$ can consist of two-dimensional compact Alexandrov spaces with a common lower (or upper) bound for the curvature. Even the very particular family of all tetrahedral surfaces seems to be quite interesting.

In the previous paper [11] the following problem was formulated.

Problem 2. Find the minimal number of triangles of a non-obtuse, respectively acute, triangulation of the Platonic surfaces in the non-trivial cases, i.e., for the surface of the cube, of the regular dodecahedron and of the regular icosahedron.

In [11] we proved together with Hangan that the surface of a cube admits several acute triangulations with 24 triangles, and no acute triangulation with fewer triangles. What about its non-obtuse triangulations?

The surface of a cube admits a non-obtuse triangulations with four triangles! Indeed, if we denote the vertices of the upper square by a, b, c, d with a and c non-adjacent, and denote the vertex of the bottom square which is adjacent to a (resp. b, c, d) by a' (resp. b', c', d'), then the four equilateral triangles bda', bdc', $a'c'b$, $a'c'd$ form a non-obtuse triangulation.

Clearly, each triangulation of any two-dimensional manifold has an even number of triangles.

In this paper we completely settle both questions about the minimal non-obtuse and the minimal acute triangulation of the regular icosahedral surface. We construct a highly asymmetrical triangulation with 12 triangles and are convinced that a (minimal) example possessing any symmetry does not exist.

2. Non-Obtuse Triangulations

Theorem 1. *The surface of the regular icosahedron admits a non-obtuse triangulation with eight triangles and no non-obtuse triangulation with fewer triangles.*

Proof. Figure 1 describes the surface of a regular icosahedron (on the left-hand side the upper half of ten equilateral triangles, on the right-hand side the lower half of ten equilateral triangles). Take a face abc of the left-hand side of Fig. 1. Let a' (resp. b', c') be the antipodal vertex of a (resp. b, c). Draw the segments from a (resp. b and c) to b', c' (resp. c', a' and a', b'). We get the non-obtuse triangles

$$abc, a'b'c', ab'c', bc'a', ca'b', a'bc, b'ca, c'ab.$$

Indeed, abc and $a'b'c'$ are equilateral and planar. In the case of the triangle $ab'c'$, there are two right angles at b' and c', while $\angle b'ac' = \pi/3$. The remaining triangles are all congruent to $ab'c'$. Hence all eight triangles are non-obtuse.

We prove now that eight is the smallest possible number of non-obtuse triangles. The only triangulations of the sphere with less than eight triangles are K_4 and the 1-skeleton of the double pyramid over a triangle. In both cases there are vertices with degree 3. If

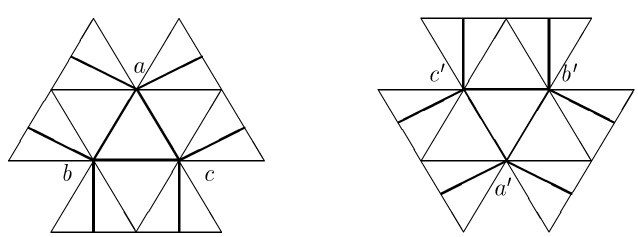

Fig. 1

the triangulation is non-obtuse, at such a vertex the total angle around it is at most $3\pi/2$. However, at each vertex of the icosahedron the total angle is $5\pi/3 > 3\pi/2$. Hence there are no non-obtuse triangulations of the icosahedron with less than eight triangles. □

3. Acute Triangulations with Few Triangles

Let $\mathcal{I}$ be the 1-skeleton of the regular icosahedron. The graph-theoretic distance $d_{\mathcal{I}}(v, w)$ between vertices v, w of $\mathcal{I}$ is called the *$\mathcal{I}$-distance*. The intrinsic distance on the surface between two points p, q is denoted by $|pq|$.

Theorem 2. *The regular icosahedral surface admits an acute triangulations with* 12 *triangles, and no acute triangulation with fewer triangles.*

Proof (*first part*). Assume the edges of the icosahedron have length 1. We present an acute triangulation consisting of 12 triangles.

First, we fix an antipodal pair of vertices a, b and a vertex a' adjacent in $\mathcal{I}$ to a. Take a vertex b' adjacent to b, at $\mathcal{I}$-distance 2 from both a and a'. Denote (consecutively) the other vertices around a by a_1, a_2, a_3, a_4. Also, denote the other vertices around b by b_1, b_2, b_3, b_4, such that b_1 is adjacent to a_2, a_3 (see Fig. 2).

Let d (resp. c') be the midpoint of the line-segment joining the midpoints of b_1a_2 (resp. a_1b_2) and b_1a_3 (resp. a_1b_3). Denote the midpoint of $b'a_4$ by x and the midpoint of a_3a_4 by y. Let c be the midpoint of the line-segment joining x with the midpoint c^+ of xy. Let z be the midpoint of b_3b_4. Take the point d' on b_4z such that $\angle d'a'c' = \pi/2$. It will be shown later that $\angle d'bb_4 < \pi/12$, $\angle bd'c' < \pi/2$, and the segments $a'b_4$ and cd' are orthogonal, whence $\angle cd'a' < \pi/2$. We choose a point d^* on $d'z$ close to d', such that $\angle d^*bb_4 < \pi/12$, $\angle bd^*c' < \pi/2$ and $\angle cd^*a' < \pi/2$ too. Note that $\angle d^*a'c' < \angle d'a'c' = \pi/2$ and $\angle d^*bc' < \angle b_4bc' = \pi/2$. Take a point c^* on $c'a_1$ close to c', such that still $\angle d^*a'c^* < \pi/2$ and $\angle d^*bc^* < \pi/2$. More conditions about how close c^* and c' must be appear later.

We got a triangulation with 12 triangles:

$$aa'c, aa'c^*, acd, ac^*d, a'cd^*, a'c^*d^*, bb'd, bb'd^*, b'cd, b'cd^*, bc^*d, bc^*d^*.$$

There are two shortest paths from b to d; here we chose the path crossing b_1b_2.

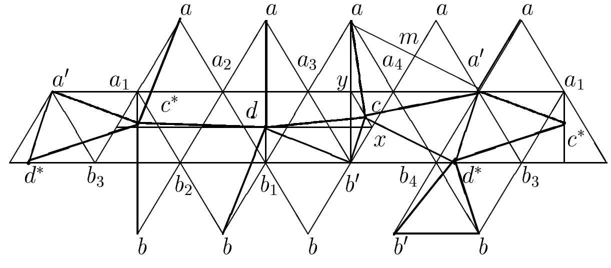

Fig. 2

We show that all these triangles are acute.

First note that, in the plane Π on which the icosahedron is unfolded in Fig. 2, the circle C of centre m and radius $\sqrt{3}/2$ passes through y. (The point $m \in \Pi$ is shown in Fig. 2.) Since, in Π, the angle xym is right, the line-segment xy is tangent to C at y. This fact yields $\angle ata' < \pi/2$ not only in Π but on the icosahedron too, for every point $t \in c^+x$. In particular, $\angle aca' < \pi/2$, $\angle ac^+a' < \pi/2$ (for any choice of the segment c^+a') and, since $\{a, c^+, a'\}$ and $\{a', c', a\}$ are congruent, $\angle a'c'a < \pi/2$ (for any choice of the segment ac'). When choosing the point c^* we arrange that $\angle a'c^*a < \pi/2$ too. Since d^* lies between z and b_4, $\angle b_4a'd^* < \pi/6$ and $\angle ca'd^* < \angle a_4a'z = \pi/2$. From the construction, $\angle d^*a'c^* < \pi/2$ and $\angle aa'c^* < \angle aa'c' < \pi/2$. Moreover, $\angle dac^* < \angle daa_1 = \pi/2$, $\angle dac < \angle daa_4 = \pi/2$, $\angle c^*aa' < \angle b_2aa' = \pi/2$, $\angle caa' < \angle b'aa' = \pi/2$ and $\angle ca'a < \angle b'a'a = \pi/2$. Around a and a' we have checked all angles.

Next we consider the angles around b and b'. It is clear that $\angle d^*b'b < \pi/3$, $\angle dbb' < \angle a_2bb' = \pi/2$, $\angle d^*bb' < \angle a'bb' = \pi/2$, $\angle dbc^* < \angle b_1bc' = \pi/2$ and $\angle db'b < \angle a_2b'b = \pi/2$, while $\angle d^*bc^* < \pi/2$ by construction.

Since $|cx|/|cy| = \frac{1}{4}$ but $|xb'|/|yb'| = 1/\sqrt{3} > \frac{1}{4}$, we have $\angle xb'c < \angle yb'c$. Thus, $\angle a_4b'c < \pi/12$. Moreover, $\angle b'ax < \angle a_4ax$. Hence

$$\angle d'a'b_4 = \angle c'a'b_2 = \angle c^+ab' < \angle b'ax < \frac{\pi}{12}.$$

Obviously, $\angle d'b'b_4 < \angle d'a'b_4$, whence $\angle d'bb_4 < \pi/12$ and, by construction, $\angle d^*bb_4 < \pi/12$ too. Hence

$$\angle cb'd^* = \angle cb'a_4 + \frac{\pi}{3} + \angle d^*b'b_4 < \frac{\pi}{2}.$$

Let p and q denote the midpoints of b_1a_3 and $b'a_3$ respectively, and $\{r\} = b'd \cap b_1a_3$ (see Fig. 3). Ceva's theorem with respect to the triangle a_3pq and the line through b' and d gives $|pr| = \frac{1}{10}$, whence $\tan \angle db'a_2 = \sqrt{3}/15$. If c^- is the orthogonal projection

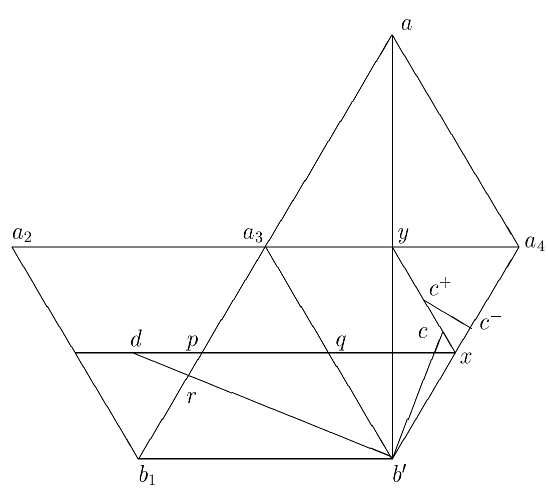

Fig. 3

of c^+ on a_4b', then $|c^+c^-| = \sqrt{3}/8$ and $|b'c^-| = \frac{5}{8}$. Thus,

$$\tan \angle a_4b'c > \frac{|c^+c^-|/2}{|b'c^-|} = \frac{\sqrt{3}}{10}.$$

Hence $\angle a_4b'c > \angle db'a_2$, which yields $\angle cb'd < \pi/2$.

Consider now the angles around d. From the choice of c^* it follows that $\angle adc^* < \angle adc' = \pi/2$. Further, $\angle adc < \angle adx = \pi/2$. Also, $\angle b'db = \angle ac'a' < \pi/2$, as shown previously. It is clear that $\angle bdc' < \pi/2$. By taking c^* close enough to c', we assure $\angle bdc^* < \pi/2$.

Obviously, $\angle b'dc$ is smaller than the angle between a_3b' and dc, which is smaller than the angle between a_3b' and b_1c, which is smaller than the right angle between a_3b' and b_1a_4.

We pass to the angles around c. We have

$$\tan \angle cdx = \frac{|cg|}{|dg|} = \frac{\sqrt{3}}{19},$$

where g is the orthogonal projection of c on dx. Also,

$$\tan \angle cab' = \frac{|cf|}{|fa|} > \frac{\sqrt{3}}{12},$$

where f is the orthogonal projection of c on $b'a$ (see Fig. 4). Hence $\angle cdx < \angle cab'$, which implies $\angle acd < \pi/2$.

We already saw that $\angle aca' < \pi/2$. It is clear that $\angle a'cd^* < \angle a'cb_4 < \pi/3$.

Let e be the intersection of the line through f and c with the perpendicular to a_4a' through d'. Simple calculations show that $|b'f| = |d'e| = 5\sqrt{3}/16$, $|fc| = \frac{3}{16}$, $|d'z| = \frac{3}{8}$

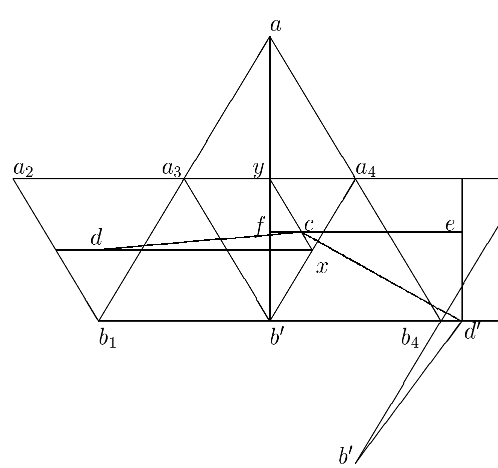

Fig. 4

and $|ce| = \frac{15}{16}$. Hence

$$\tan \angle cb'f = \frac{\sqrt{3}}{5} < \frac{\sqrt{3}}{3} = \tan \angle ecd',$$

which implies $\angle b'cd' < \pi/2$. It is clear that $\angle b'cd < \pi/2$.

Now consider the angles around c^*. The angle ac^*a' was shown to be acute. Also, $\angle ac'd < \angle a_1cd = \pi/2$, and if c^* is close enough to c', then $\angle ac^*d < \pi/2$ too. Clearly, $\angle dc^*b < \angle dc'b = \pi/2$ and

$$\angle bc^*d^* < \angle bc'd^* < \angle bc'b_4 < \frac{\pi}{2}.$$

Because $|a'c^*| > |a'a_1| > |a'd^*|$, we have $\angle a'd^*c^* > \angle a'c^*d^*$, whence $\angle a'c^*d^* < \pi/2$.

Finally, we consider the angles around d^*. It is clear that $\angle a'd^*c^* < \angle a'd^*b_3 < \pi/2$. We found above that $\tan \angle ecd' = \sqrt{3}/3$, which means that $\angle ecd' = \pi/6$. Hence $d'c$ is orthogonal to $a'b_4$; therefore $\angle b'd'c < \pi/2$. Also, $\angle b'd^*c < \angle b'd'c < \pi/2$. It is obvious that $\angle bd^*b' < \angle bb_4b' = \pi/3$.

Let v, w denote the midpoints of b_3a_1, b_3a' respectively (see Fig. 5). We already found $|d'z| = \frac{3}{8}$. Hence

$$|b_4d'| = \tfrac{1}{8} < \tfrac{1}{4} = |vc'|.$$

It follows that the angle α between b_4w and $d'c'$ is larger than $\angle wb_4v$. We also already found that $\tan \angle db'a_2 = \sqrt{3}/15$. Moreover,

$$\angle db'a_2 = \angle c'a'v = \angle b_4a'd' = \angle b_4bd'$$

and $\tan \angle wb_4v = \sqrt{3}/9$. So,

$$\alpha > \angle wb_4v > \angle db'a_2 = \angle b_4bd'.$$

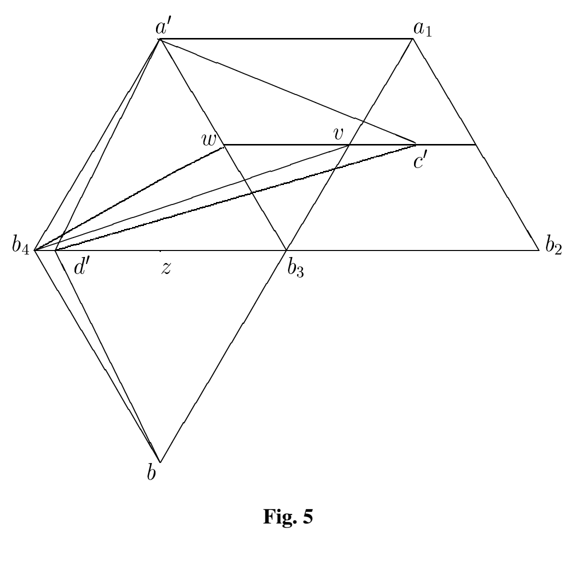

Fig. 5

Hence $\angle bd'c' < \pi/2$. Now, our initial construction guarantees that $\angle bd^*c' < \pi/2$ too. Finally, by construction, $\angle cd^*a' < \pi/2$.

We finished the proof that all triangles are acute. □

4. No Acute Triangulation with Even Fewer Triangles

We start by proving the following lemma.

Lemma 1. *Suppose a, b, b' are vertices of the acute triangulation $\mathcal{T}$ of the regular icosahedral surface, such that a has degree 4 in $\mathcal{T}$, bb' is an edge of $\mathcal{I}$ but not of $\mathcal{T}$ and b, b' are neighbours of a in $\mathcal{T}$. Then $\angle abb' > 2\pi/3$ and $\angle ab'b > 2\pi/3$.*

Proof. The above vertices b, a, b' of $\mathcal{T}$ form a path of length 2 in $\mathcal{T}$.

Since the degree of a in $\mathcal{T}$ is 4, a is a vertex of $\mathcal{I}$. Because ab and ab' are non-consecutive edges of $\mathcal{T}$ at a and $\mathcal{T}$ is acute, necessarily $\angle bab' > 2\pi/3$.

After fixing the edge bb', we look for the possible location of a.

If $d_{\mathcal{I}}(a, b) = d_{\mathcal{I}}(a, b') = 1$, then $\angle bab' = \pi/3$, which is bad. If $d_{\mathcal{I}}(a, b) = 2$ and $d_{\mathcal{I}}(a, b') = 1$, then $\angle bab' = \pi/6$, even worse. If $d_{\mathcal{I}}(a, b) = d_{\mathcal{I}}(a, b') = 2$, then $\angle bab' = \pi/3$, bad again.

The only remaining case (essentially) is $d_{\mathcal{I}}(a, b) = 3$ and $d_{\mathcal{I}}(a, b') = 2$, when a and b are opposite vertices of the icosahedron. Looking at Fig. 2 we find points a, b, b' in such a position. There are ten segments from a to b. By choosing one of those two closest to a_1 as ab, we indeed have $\angle bab' > 2\pi/3$, for all the other eight this inequality fails to be true. Then $\angle abb' > 2\pi/3$ and $\angle ab'b = 5\pi/6 > 2\pi/3$ too. □

Proof of Theorem 2 (*last part*). By Theorem 1, we only have to show that there is no acute triangulation of the regular icosahedral surface with eight or ten triangles.

Suppose there exists an acute triangulation $\mathcal{T}$ of the regular icosahedral surface containing eight triangles. The only triangulation of the sphere with eight triangles and degree at least 4 at every vertex is the 1-skeleton of the regular octhedron. All its vertices have degree 4. Therefore all vertices of $\mathcal{T}$ are vertices of $\mathcal{I}$.

It is easily seen that any acute angle pqr between two segments pq and qr joining vertices of the icosahedron is at most $\pi/3$ or equals $\angle bb_1x$ or equals $\angle ab_1x$ (see Fig. 2). The second and third values appear only if q is antipodal either to p or to r.

Let p_0 be a vertex of $\mathcal{T}$. Because the triangulation is acute, among the four angles around p_0 at most one can be equal to or less than $\pi/3$. Further, at most two among the angles around p_0 can take the second or third value mentioned above, because at most one of the four distinct neighbours of p_0 can be antipodal to p_0. Since there is a fourth angle at p_0, a contradiction is obtained.

Suppose now that a triangulation $\mathcal{T}$ with ten acute triangles exists. The only triangulation of the sphere with ten triangles and degree at least 4 at every vertex is the 1-skeleton of the double pyramid over the pentagon. Let C_5 be the 5-cycle in $\mathcal{T}$ containing all 4-valent vertices. Clearly, the vertices of C_5 must be vertices of $\mathcal{I}$.

We claim that there are two vertices p_1, p_2 of C_5 such that $p_1 p_2$ is an edge of $\mathcal{I}$.

Indeed, suppose this is not true. Then all five neighbours in $\mathcal{I}$ of $p_1 \in C_5$ do not belong to C_5. Among all five vertices at $\mathcal{I}$-distance 2 from p_1 there are at most two vertices in C_5. Now, only one vertex is left, while C_5 contains five vertices! So, the claim is proved.

We further claim that there are three vertices $p_1, p_2, p_3 \in C_5$ such that $p_1 p_2$ and $p_2 p_3$ are edges of $\mathcal{I}$. Indeed, we already found an edge $p_1 p_2$ of $\mathcal{I}$ with $p_1, p_2 \in C_5$, but suppose the new claim is wrong. Then the six vertices of $\mathcal{I}$ at $\mathcal{I}$-distance 1 from $\{p_1, p_2\}$ are not in C_5. So the remaining K_4 minus one edge must contain the other three vertices of C_5, which is impossible without contradicting the assumption. Also the second claim is true.

Clearly, $p_1 p_2$ and $p_2 p_3$ cannot both be edges of $\mathcal{T}$, because if they were, then $\angle p_1 p_2 p_3 \leq 2\pi/3$ and $\mathcal{T}$ would not be acute.

The cycle C_5 contains two more vertices, p_4 and p_5. So there are (essentially) two cases concerning the order on C_5: p_1, p_4, p_2, p_5, p_3 and p_1, p_2, p_4, p_3, p_5. Also, $d_{\mathcal{I}}(p_1, p_3)$ may be 1 or 2.

Case p_1, p_4, p_2, p_5, p_3. The presence of the segment $p_1 p_3$ implies

$$\angle p_1 p_2 p_4 + \angle p_4 p_2 p_5 + \angle p_5 p_2 p_3 \leq \frac{4\pi}{3}.$$

(If $d_{\mathcal{I}}(p_1, p_3) = 1$, the sum is $4\pi/3$, if $d_{\mathcal{I}}(p_1, p_3) = 2$, the sum is π.) However, $\angle p_1 p_2 p_4 > 2\pi/3$ and $\angle p_5 p_2 p_3 > 2\pi/3$ by Lemma 1, and a contradiction is obtained.

Case p_1, p_2, p_4, p_3, p_5. We have

$$\angle p_1 p_2 p_4 + \angle p_4 p_2 p_3 \leq \frac{4\pi}{3}.$$

(If $d_{\mathcal{I}}(p_1, p_3) = 1$, the sum is $\pi/3$ or $4\pi/3$, if $d_{\mathcal{I}}(p_1, p_3) = 2$, the sum is $2\pi/3$ or π.) However, $\angle p_1 p_2 p_4 > 2\pi/3$ because $\mathcal{T}$ is acute, and $\angle p_4 p_2 p_3 > 2\pi/3$ by Lemma 1. A contradiction is obtained again. □

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