

EXTENDING STECHKIN'S THEOREM AND BEYOND

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We generalize Stechkin's theorem about the single-valuedness of the nearest point mapping at most points to any complete length space without bifurcating geodesics and make further observations.

Let $K \subset \mathbb{R}^d$ be a closed set and p_K the *nearest point mapping*, which associates with every point $x \in \mathbb{R}^d$ the set of all points in K closest to x . It is well known that p_K is single-valued a.e. The topological counterpart was provided by Stechkin [6] in 1963, who proved that p_K is single-valued at *most* points of \mathbb{R}^d , that is, the set of points at which p_K is not single-valued is of the first Baire category.

A result attributed to Konjagin says that the set of points where p_K is not single-valued is even σ -porous (see also [7]). This implies both statements, on measure and category.

We showed in [8] that, in any Alexandrov space with curvature bounded below, p_K is properly multivalued on a σ -porous set.

We extend here Stechkin's result to more general metric spaces, and provide related results under even more general assumptions.

Let (X, ρ) be a metric space and let $K \subset X$ be compact. As usual, for $x \in X$,

$$p_K(x) = \{y \in K : \rho(x, y) = \rho(x, K)\}, \quad (1)$$

where $\rho(x, K) = \min_{z \in K} \rho(x, z)$.

The open ball of centre x and radius ε will be denoted by $B(x, \varepsilon)$. We will also use the notation

$$A_n = \left\{ x \in X : \text{diam } p_K(x) \geq \frac{1}{n} \right\}. \quad (2)$$

Let \mathcal{H} be the space of all nonempty compact subsets of X and H the Pompeiu-Hausdorff distance in \mathcal{H} , that is,

$$H(K, K') = \max \left\{ \max_{a \in K} \rho(a, K'), \max_{b \in K'} \rho(b, K) \right\}. \quad (3)$$

The space (\mathcal{H}, H) is complete as soon as (X, ρ) is itself complete.

An arc σ in X from x to y is called a *segment* if its length $\lambda\sigma$ equals $\rho(x, y)$. A metric space (with more than one point) in which there is a segment between any two points is called a *length space*.

We say that a metric space is *without bifurcating geodesics* if, for any two segments starting at the same point and having another common point, this second point is a common endpoint of both or one segment includes the other.

We start with the following generalization of Stechkin's theorem.

THEOREM 1. *For any compact set K in the complete length space (X, ρ) without bifurcating geodesics, p_K is single-valued at most points $x \in X$.*

Proof. Suppose the conclusion of the theorem is false. Then p_K is not single-valued on a set of second category. This means that $\bigcup_{n=1}^{\infty} A_n$ is of second category. Hence, for some index n , A_n is dense in a ball $B(a, r)$.

Let $y \in p_K(a)$. Consider a segment ay and some point $x \in ay \cap B(a, r)$. Let $y' \in p_K(x)$. Since $\rho(a, y') \geq \rho(a, y)$ and

$$\rho(a, y') \leq \rho(a, x) + \rho(x, y') \leq \rho(a, y), \tag{4}$$

we must have the equality sign, and the segments ay and $ax \cup xy'$ would bifurcate if $y \neq y'$. It follows that $p_K(x) = \{y\}$.

Since p_K is upper semicontinuous, there is some neighbourhood N of x such that $p_K(u) \subset B(y, 1/(3n))$ for all $u \in N$.

Then $\text{diam } p_K(u) < 1/n$ for all these u , A_n is not dense in $B(a, r)$, and a contradiction is obtained.

Hence the conclusion of the theorem is true. □

Theorem 1 shows that p_K is single-valued at many points. It is, however, rarely single-valued everywhere.

THEOREM 2. *In any length space (X, ρ) , p_K is single-valued only for a nowhere dense family of sets $K \in \mathcal{K}$.*

Proof. Let $\mathcal{O} \subset \mathcal{K}$ be open and assume p_K is single-valued for some set $K \in \mathcal{O}$. If $K \neq X$, take $x \in X \setminus K$, put $\{z_2\} = p_K(x)$, and take $z_1 \in xz_2$ at distance ε from z_2 such that $K' = K \cup \{z_1\} \in \mathcal{O}$. If $K = X$, take a segment $\sigma \subset X$, choose its midpoint z , and consider a positive number $\varepsilon \leq \lambda\sigma$ such that every $K^* \in \mathcal{K}$ with $H(K^*, X) < \varepsilon$ belongs to \mathcal{O} . Now let z_1, z_2 denote the two points of σ at distance $\varepsilon/2$ from z and define

$$K' = (X \setminus B(z, \varepsilon)) \cup \{z_1, z_2\}. \tag{5}$$

Clearly, $K' \in \mathcal{O}$.

If $H(K', K'') < \varepsilon/4$, $p_{K''}$ is not single-valued. Indeed, there are points $u_1, u_2 \in K''$ with $\rho(u_i, z_i) < \varepsilon/4$. Of course, $p_{K''}(z_i) \in B(z_i, \varepsilon/4)$ for both i . Also, for every point x on the segment z_1z_2 ,

$$p_{K''}(x) \in B\left(z_1, \frac{\varepsilon}{4}\right) \cup B\left(z_2, \frac{\varepsilon}{4}\right). \tag{6}$$

It follows that there must be a point $u \in z_1 z_2$ with $p_{K''}(u)$ properly multivalued. Since $\{K'' : H(K', K'') < \varepsilon/4\}$ is open in \mathcal{K} , the theorem is proven. \square

We presented in [8, Theorem 3] a result on most compact sets in an Alexandrov space of curvature bounded below. Since the only property exploited in its proof is the absence of bifurcating geodesics, we can formulate the following theorem and refer to the proof in [8].

THEOREM 3. *In a complete separable length space (X, ρ) without bifurcating geodesics, in which every open set has Hausdorff dimension larger than 1, for most $K \in \mathcal{K}$, p_K is properly multivalued at a dense set of points.*

Already [8, Theorem 3] is a generalization of an old result of the author [7], which had been repeatedly strengthened in various ways, for example, in [1, 2, 3, 4, 5, 9].

Without the condition about the nonexistence of bifurcating geodesics, we can only prove the following theorem.

THEOREM 4. *For most compact sets K in the complete separable length space (X, ρ) , p_K is single-valued at most points $x \in X$.*

Proof. Fix an open ball $B(a, r) \subset X$. We show that the set \mathcal{K}_n of all compact sets $K \subset X$ for which $\{x \in X : \text{diam } p_K(x) \geq 1/n\}$ is dense in $B(a, r)$ is nowhere dense in \mathcal{K} .

Let $\mathcal{O} \subset \mathcal{K}$ be open. We choose $K \in \mathcal{O}$ such that $a \notin K$.

Let $y \in p_K(a)$. Choose a segment ay and a point $z \in ay$ such that $K \cup \{z\} \in \mathcal{O}$. Take

$$\varepsilon = \min \left\{ \frac{\rho(y, z)}{4}, \frac{1}{2n} \right\}. \tag{7}$$

Then, for every point $x \in B(a, \varepsilon)$, every point v at distance at most ε from K and every point $w \in B(z, \varepsilon)$, we have

$$\begin{aligned} \rho(x, v) &> \rho(a, y) - 2\varepsilon, \\ \rho(x, w) &< \rho(a, z) + 2\varepsilon. \end{aligned} \tag{8}$$

Hence, for every point $x \in B(a, \varepsilon)$ and every compact set K' at Pompeiu-Hausdorff distance at most ε from $K \cup \{z\}$,

$$p_{K'}(x) \subset B(z, \varepsilon), \tag{9}$$

whence

$$\text{diam } p_{K'}(x) < \frac{1}{n}. \tag{10}$$

Let K be such that p_K is not single-valued on a set of second category. Then

$$\{x \in X : \text{diam } p_K(x) > 0\} = \bigcup_{n=1}^{\infty} A_n \tag{11}$$

is of second category. This implies that, for some n , the set A_n is dense in some ball.

Since X is separable, there is a countable set $\{x_i\}_{i=1}^{\infty}$ dense in X . Hence A_n is dense in some ball $B(x_i, r_j)$ with rational radius r_j .

Hence all compact sets K for which p_K is not single-valued on a set of second category belongs to the union $\bigcup_{n,i,j=1}^{\infty} \mathcal{H}_{n,i,j}$, where $\mathcal{H}_{n,i,j}$ is the set of those $K \in \mathcal{K}$ for which A_n is dense in $B(x_i, r_j)$.

Since we showed that each $\mathcal{H}_{n,i,j}$ is nowhere dense, the union above is of first category and the theorem is proven. \square

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