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Simplices passing through a hole

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Abstract. We study small holes through which regular 3-, 4-, and 5-dimensional simplices can pass.

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Assume a plane $P \subset \mathbb{R}^3$ has a hole H, and that the hole is topologically a compact disc. Being so, $P \setminus H$ does not separate the space. A regular tetrahedron σ^3 (of edge-length 1, say) wants to pass through the hole. Can it? Obviously, it can't if H is circular of diameter less than the width of a face of σ^3, namely $\sqrt{3}/2$. And it obviously can if the diameter of H, again supposed circular, is at least $2/\sqrt{3}$.

If the hole H is a disc of diameter just slightly larger than $\sqrt{2}/2$, then of course σ^3 cannot pass through the hole. But if it was born there? There are positions of σ^3 from which, once trapped, it cannot escape from the plane. See [1] for details.

Less obviously, if the hole is a disc of diameter $\sqrt{3}/2$ or slightly larger, then the tetrahedron still cannot pass through it.

A hole of width $\sqrt{2}/2$ through which σ^3 can pass can easily be found, for example when the hole is square in shape. This hole will have diameter 1.

We look in this paper for a convex hole H of diameter $\sqrt{3}/2$, the width of a face of σ^3, and of width $\sqrt{2}/2$, the width of σ^3, such that σ^3 can pass through H.

Let σ^n be an n-dimensional regular simplex of edge-length 1 and of variable position in $\mathbb{R}^n$, and $P = \{(x_1, \ldots, x_{n-1}, 0) : x_1, \ldots, x_{n-1} \in \mathbb{R}\}$ the hyperplane which will contain the hole. We'll often drop the n-th coordinate when points in P are considered.

THEOREM A. *There exists a convex hole $H \subset P$ of diameter $\frac{\sqrt{3}}{2}$ and width $\frac{\sqrt{2}}{2}$ such that the regular tetrahedron σ^3 moving in $\mathbb{R}^3$ can pass through H.*

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Proof. Take a square $Q \subset P$ of edge-length $\frac{1}{2}$, with vertices $q_{\pm,+} = (\pm\frac{1}{4}, \frac{1}{2})$ and $q_{\pm,-} = (\pm\frac{1}{4}, 0)$. Denote the point $(0, \frac{\sqrt{11}}{4}) \in P$ by v. Take a disc D of centre $(0, \frac{\sqrt{2}}{4})$ and radius $\frac{\sqrt{2}}{4}$. Define the hole H as the convex hull of $D \cup T$, where T is the triangle $vq_{+-}q_{--}$.

Let $p_1, \dots, p_4$ be the (variable) vertices of σ^3. Let $m_{i,j}$ be the midpoint of the edge $p_i p_j$ of σ^3. We show how σ^3 can pass through H from the upper half-space of $\mathbb{R}^3$ to the lower half-space.

First we move σ^3 toward H, let the vertex p_1 cross H and continue until the point $m_{1,3}$ comes to q_{+-} and $m_{1,4}$ to q_{--}. Rotate σ^3 around the line $q_{+-}q_{--}$ such that p_2 moves toward P. Continuing the rotation, p_2 goes through v and we stop rotating when we get the position of σ^3 with $m_{1,3}, m_{1,4}, m_{2,4}, m_{2,3}$ at the vertices of Q. Translate σ^3 by the vector $(0, \frac{\sqrt{2}-1}{4}, 0)$. The centre of the square $m_{1,3}m_{1,4}m_{2,4}m_{2,3}$ now coincides with the centre of D. Rotate σ^3, say clockwise, by an angle $\pi/2$ around the line through the centre of D, orthogonal to P. Translate σ^3 by the vector $(0, -\frac{\sqrt{2}-1}{4}, 0)$. We get a position of σ^3 with the point $m_{1,4}$ at q_{+-} and the point $m_{2,4}$ at q_{--}. Rotate again σ^3 around the line $q_{+-}q_{--}$. Then p_3 goes through v and the rest is just a translation downwards. Hence σ^3 passed through H. □

We already mentioned that the above values for the diameter and width are best possible.

Next we consider the case of the 4-dimensional regular simplex σ^4 in the 4-dimensional Euclidean space.

THEOREM B. *There is a convex hole $H \subset P$ of diameter $\frac{\sqrt{3}}{2}$ and width less than $\frac{\sqrt{2}}{2}$ such that the regular simplex σ^4 moving in $\mathbb{R}^4$ can pass through H.*

Proof. Take a prism $R \subset P$ whose vertices $r_1, r_2, r_3, r_1', r_2', r_3'$ are given as follows.

$$r_1 = \left(\frac{\sqrt{3}}{6}, 0, 0\right), \; r_2 = \left(-\frac{\sqrt{3}}{12}, \frac{1}{4}, 0\right), \; r_3 = \left(-\frac{\sqrt{3}}{12}, -\frac{1}{4}, 0\right),$$

$$r_1' = \left(\frac{\sqrt{3}}{6}, 0, \frac{1}{2}\right), \; r_2' = \left(-\frac{\sqrt{3}}{12}, \frac{1}{4}, \frac{1}{2}\right), \; r_3' = \left(-\frac{\sqrt{3}}{12}, -\frac{1}{4}, \frac{1}{2}\right).$$

Denote the point $(0, 0, \frac{\sqrt{6}}{3}) \in P$ by v. Let $q_{\pm,\pm}$ be the point $(\pm\frac{1}{4}, \pm\frac{1}{4}, a)$, where $a = \frac{\sqrt{6}}{3} - \frac{\sqrt{10}}{4}$. Take a disc Δ in the plane $x_3 = a$, of centre $(0, 0, a)$ and radius $\frac{\sqrt{2}}{4}$. Let Q be the cone with apex v and based on Δ. Note that $q_{\pm,\pm}$ are on the boundary circle of Δ and

$$\|v - r_i\| = \|v - q_{\pm,\pm}\| = \frac{\sqrt{3}}{2}.$$

Take a cylinder D such that the base discs of D are contained in the 2-dimensional planes $x_2 = \pm\frac{1}{4}$, their centres are $(0, \pm\frac{1}{4}, \frac{\sqrt{7}}{8})$ and their radii are equal to $\frac{\sqrt{7}}{8}$. Define the hole H to be the convex hull of $R \cup Q \cup D$. (Note that the prism R' with vertices $q_{\pm,\pm}$ and $(\pm\frac{1}{4}, 0, a + \frac{\sqrt{3}}{4})$ is contained in H.)

Let $p_1, \dots, p_5$ be the vertices of σ^4 and $m_{i,j}$ the midpoint of the edge $p_i p_j$ of σ^4. We show how σ^4 can pass through H from the upper half-space of $\mathbb{R}^4$ to the lower half-space.

STEP 1. First we move σ^4 toward H, let the vertex p_1 cross H and continue until the point $m_{1,3}$ comes to r_1, $m_{1,4}$ comes to r_2, $m_{1,5}$ comes to r_3, and $m_{1,2}$ also takes a position on P, too. Then $\sigma^4 \cap P$ is a regular tetrahedron. Rotate σ^4 around the plane through $m_{1,3}$, $m_{1,4}$ and $m_{1,5}$ such that p_2 moves toward P. Continuing the rotation, p_2 goes through v, and we stop rotating when we get the position of σ^4 with $m_{i,j}$ on P ($i = 1, 2$, $j = 3, 4, 5$), that is, with $m_{1,3}, m_{1,4}, m_{1,5}$ still at r_1, r_2, r_3 and $m_{2,3}, m_{2,4}, m_{2,5}$ at r_1', r_2', r_3' respectively. Note that $\sigma^4 \cap P$ becomes now the prism R. Translate σ^4 by the vector $w = (-\frac{\sqrt{3}}{24}, 0, \frac{\sqrt{7}}{8} - \frac{1}{4}, 0)$, so $\sigma^4 \cap P$ becomes the prism $R + w$.

The baricentre of the prism does not coincide with the centre of D, but now $R + w$ is inscribed in D. Rotate σ^4 around the axis of D with a rotation angle of $\frac{\pi}{2}$ in such a way that the new position of the edge $m_{1,3}m_{2,3}$ lies between v and the square face $m_{1,4}m_{2,4}m_{2,5}m_{1,5}$. Translate σ^4 by the vector $(0, 0, -\frac{\sqrt{7}}{8} - \frac{1}{4} + a, 0)$. We get *a* new position for $\sigma^4 \cap P$, as the point $m_{1,4}$ comes to q_{++}, $m_{2,4}$ to q_{-+}, $m_{1,5}$ to q_{+-}, $m_{2,5}$ to q_{--}, $m_{1,3}$ to $(\frac{1}{4}, 0, a + \frac{\sqrt{3}}{4})$ and $m_{2,3}$ to $(-\frac{1}{4}, 0, a + \frac{\sqrt{3}}{4})$. We just positioned σ^4 to let its third vertex pass through H.

STEP 2. Rotate σ^4 around the plane $x_3 = a$, $x_4 = 0$, keeping the square $C = m_{1,4}m_{2,4}m_{2,5}m_{1,5}$ fixed. Then the vertex p_3 arrives precisely at the point $v \in H$.

If we continued the rotation, $\sigma^4 \cap P$ would become another prism with the same square C as a face, but with the points $m_{3,4}$ and $m_{3,5}$, new on P, at $(0, \frac{1}{4}, a + \frac{\sqrt{3}}{4})$ and $(0, -\frac{1}{4}, a + \frac{\sqrt{3}}{4})$ respectively. Instead, at this stage, we rotate σ^4 around the plane $x_1 = x_2 = 0$ by an angle of $\pi/2$ again.

After this, we can perform our rotation around C, and come out with σ^4 from the hole by following the same steps in inversed order. □

We will now consider the 5-dimensional case.

THEOREM C. *There is a convex hole* $H \subset P$ *with diameter* $\frac{\sqrt{3}}{2}$ *and width* $\frac{\sqrt{6}}{3}$, *such that the regular simplex* σ^5 *moving in* $\mathbb{R}^5$ *can pass through* H.

Proof. Take the regular tetrahedron $\Delta^3 \subset P$ with edge-length $\frac{1}{2}$ and vertices at

$$q_1 = \left(0, 0, \frac{\sqrt{6}}{6} - \frac{\sqrt{6}}{24}, 0, 0\right), \ q_2 = \left(-\frac{\sqrt{3}}{6}, 0, -\frac{\sqrt{6}}{24}, 0, 0\right),$$

$$q_3 = \left(\frac{\sqrt{3}}{12}, \frac{1}{4}, -\frac{\sqrt{6}}{24}, 0, 0\right), \ q_4 = \left(\frac{\sqrt{3}}{12}, -\frac{1}{4}, -\frac{\sqrt{6}}{24}, 0, 0\right),$$

and consider the triangle Δ^2 with vertices q_1, q_2, q_3.

Now, for $j = 2, 3$, take a regular $(4-j)$−dimensional simplex Δ_{4-j} as follows. For $j = 3$ consider the line-segment Δ_1 on the x_4-axis, with endpoints

$$q_1^3 = (0, 0, 0, 0, 0), \ q_2^3 = \left(0, 0, 0, \frac{1}{2}, 0\right).$$

For $j = 2$, consider the triangle Δ_2 with vertices

$$q_1^2 = \left(0, 0, -\frac{\sqrt{6}}{24}, 0, 0\right), \ q_2^2 = \left(0, 0, -\frac{\sqrt{6}}{24}, 0, 0\right), \ q_3^2 = \left(0, 0, \frac{\sqrt{3}}{4} - \frac{\sqrt{6}}{24}, \frac{1}{4}, 0\right).$$

Let $x^k(q^l)$ (resp. $x^k(q_m^j)$) be the x_k-coordinate of q_l (resp. q_m^j). Denote the point $(x^1(q^l), x^2(q^l), x^3(q^l), x^4(q_m^3), 0)$ by (q^l, q_m^3), where $l = 1, 2, 3, 4$, $m = 1, 2$ and denote the point $(x^1(q^l), x^2(q^l), x^3(q_m^2), x^4(q_m^2), 0)$ by (q^l, q_m^2), where $l = 2, 3, 4$, $m = 1, 2, 3$. Note that the convex hull of $\{(q^l, q_m^3) : l \in \{1, 2, 3, 4\}, m \in \{1, 2\}\}$ is the Cartesian product $\Delta^3 \times \Delta_1$ and the convex hull of $\{(q^l, q_m^2) : l \in \{2, 3, 4\}, m \in \{1, 2, 3\}\}$ is the Cartesian product $\Delta^2 \times \Delta_2$. Note that $(q^l, q_1^3) = (q^l, q_1^2)$ and $(q^l, q_2^3) = (q^l, q_2^2)$, if $l = 2, 3, 4$.

Denote the point $(0, 0, 0, \frac{\sqrt{21}}{\sqrt{32}}, 0) \in P$ by v. Let D be the sphere of centre $(0, \ldots, 0, \frac{\sqrt{2}}{2\sqrt{3}})$ and radius $\frac{\sqrt{2}}{2\sqrt{3}}$. Note that $\|(q^l, q_1^3)\| = \frac{\sqrt{3}}{2}$, if $l = 2, 3, 4$, $\|v - (q^1, q_1^3)\| < \frac{\sqrt{3}}{2}$ and $\|v - (q^l, q_m^1)\| < \frac{\sqrt{3}}{2}$. Elementary calculation yields

$$\|v - (q^l, q_3^2)\| = \sqrt{\frac{95}{96} - \frac{\sqrt{2}}{16} - \frac{1}{2}\sqrt{\frac{21}{32}}} < \frac{\sqrt{3}}{2}.$$

Define the hole H as the convex hull of $(\Delta^3 \times \Delta_1) \cup (\Delta^2 \times \Delta_2) \cup D \cup \{v\}$. Note that the diameter of H is $\frac{\sqrt{3}}{2}$ and the width of H is $\frac{\sqrt{6}}{3}$.

Let $p_1, \ldots, p_6$ be the vertices of σ^5. As before, let $m_{i,j}$ be the midpoint of $p_i p_j$.

We show now how σ^5 can pass through H from the upper half-space to the lower half-space of $\mathbb{R}^5$.

First, we move σ^5 toward H, let the vertex p_1 cross H, continue such that the point $m_{1,k}$ ($k = 3, \ldots, 6$) comes to q_{k-2}, and put $m_{1,2}$ also on P. Rotate σ^5 around the 3-dimensional subspace spanned by $\{m_{1,3}, \ldots, m_{1,6}\}$, such that p_2 moves toward P. Continuing the rotation, p_2 goes through v and we stop rotating when we σ^5 has

$$m_{1,3}, \ldots, m_{1,6}, m_{2,3}, \ldots, m_{2,6}$$

contained in P. Then the section $\sigma^5 \cap P$ coincides with $\Delta^3 \times \Delta_1$, as

$$m_{1,k} = (q^{k-2}, q_1^3), \quad m_{2,k} = (q^{k-2}, q_2^3) \quad (k = 3, \ldots, 6).$$

Next, rotate σ^5 around the 3-dimensional affine subspace $x_3 = 0$ of P spanned by

$$\{m_{1,4}, m_{1,5}, m_{1,6}, m_{2,4}, m_{2,5}, m_{2,6}\},$$

such that p_3 moves toward P. Continuing the rotation, p_3 goes through v and we eventually get a position of σ^5 with the midpoints

$$m_{1,4}, m_{1,5}, m_{1,6}, m_{2,4}, m_{2,5}, m_{2,6}, m_{3,4}, m_{3,5}, m_{3,6}$$

contained in P. Now $\sigma^5 \cap P$ coincides with $\Delta^2 \times \Delta_2$, as $m_{1,k} = (q^{k-2}, q_1^2)$, $m_{2,k} = (q^{k-2}, q_2^2)$, $m_{3,k} = (q^{k-2}, q_3^2)$ ($k = 4, 5, 6$). Note that p_1, p_2, p_3 already passed through H.

Translate σ^5 in the direction $(0, 0, 0, 1, 0)$ until the centre of $\sigma^5 \cap P$ coincides with the centre of D. Note that the section $\sigma^5 \cap P$ is a product of two equilateral triangles of edge-length $\frac{1}{2}$. Rotate σ^5 leaving P invariant and exchanging the two triangles such that when we translate σ^5 in direction $(0, 0, 0, -1, 0)$, we get

$$m_{1,k} = (q^1, q_{k-3}^3), \quad m_{2,k} = (q^2, q_{k-3}^3), \quad m_{3,k} = (q^3, q_{k-3}^3),$$

for $k = 4, 5, 6$.

Now we have a position for σ^5 with $m_{l,6} = (q^l, q_3^3)$ for $l = 1, 2, 3$. It only remains to move σ^5 as above, but in converse order. So can σ^5 pass through H. □

PROBLEM. *Find the minimal $(n-1)$-dimensional volume of a compact hole in a hyperplane of $\mathbb{R}^n$ such that the regular simplex σ^n of edge-length 1 can pass through it.*

The answer to this problem is not clear even for $n = 3$.

We know that the hole H constructed in the proof of Theorem A has not the smallest possible area. For example, using the proof of Theorem A and its notation, it can be immediately seen that $Q \cup D \cup T$ is a suitable nonconvex hole included in H; and there are further improvements in addition to this.

REMARK. K. Zindler [2] considered already in 1920 a convex polytope which can pass through a fairly small circular hole. He proved the intriguing fact that the smallest infinite circular cylinder containing the polytope has its circular section larger than the hole.

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