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Symmetry and the farthest point mapping on convex surfaces

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Abstract. Consider the mapping F associating to each point x of a convex surface the set of all points at maximal intrinsic distance from x. We provide a large class of surfaces on which F is single-valued and involutive. Moreover, we show that there are point-symmetric surfaces of revolution with F single-valued but not involutive.

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Introduction

Let $\mathscr{S}$ be the space of all closed convex surfaces (i.e. boundaries of open bounded convex sets) in the 3-dimensional Euclidean space, endowed with the usual Pompeiu–Hausdorff metric.

Denote by F_x the set of farthest points from x (absolute maxima of the intrinsic distance from x) and by F the *farthest point mapping*, i.e., the multivalued mapping associating to $x \in S$ the set F_x.

Our results are related to a conjecture of Steinhaus saying that if on a convex surface S the mapping F is single-valued and $F \circ F = \mathrm{id}_S$ (F is an involution) then the surface is a sphere ([3], p. 44). This conjecture was disproved by the first author [10]. He constructed a large family $\mathscr{R}$ of convex surfaces with both axial and central symmetry, on which F is single-valued and involutive (with $F_x = \{-x\}$). Then the following question naturally arose [10]. Is it true, for convex surfaces on which F is single-valued, that F is involutive? Or is this at least true for point-symmetric convex surfaces of revolution?

On the (point-symmetric) boundary K of a 1-times-1-times-2 box (recall the Knuth–Kotani puzzle), the mapping F, even restricted to the vertex set, is not single-valued. In this paper we see that suitable bounds on the curvature of S, or on curvature and radius, guarantee the farthest point mapping to be a homeomorphism. Nevertheless, the answer to the preceding questions will be shown to be negative. We also extend the family $\mathscr{R}$ given in [10] to the family $\mathscr{I}$ of all convex surfaces of

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constant intrinsic radius, and we characterize the centrally symmetric surfaces which verify Steinhaus' conditions.

In the last years, several questions about farthest points proposed by H. Steinhaus (see the chapter A35 of the book [3] by H. T. Croft, K. J. Falconer and R. K. Guy) have been answered by the second author (see [12], [14], [15]).

It is now known, for example, that on any convex surface S, for almost all points $x \in S$ (in the sense of measure), F_x contains a single point [15]. J. Rouyer [7] showed that similar results hold true (in the framework of Riemannian geometry) for surfaces which are homeomorphic to the 2-sphere, but not necessarily convex.

1 Definitions and notation

For $S \in \mathscr{S}$ and $x, y \in S$, $\rho(x, y)$ will be the geodesic (intrinsic) distance between x and y, and ρ_x the distance function from x: $\rho_x(y) = \rho(x, y)$.

Also, let C_x be the set of all points in S joined to x by at least two *segments*, i.e. shortest paths, M_x be the set of all relative maxima for ρ_x, and $C(x)$ be the *cut locus* of x, i.e., the set of all points $y \in S$ such that some segment from x to y is not extendable as a segment beyond y. Of course, $C_x \subset C(x)$ and $M_x \subset C(x)$.

The 1-dimensional Hausdorff measure (length) of the set $A \subset S$ is denoted by λA, while card A denotes its cardinality.

For $x \in S$, let T_x denote the space of tangent directions at x; it can be regarded as a closed curve, the intersection of the tangent cone at x with the 2-dimensional unit sphere. Thus, $\lambda T_x \leqslant 2\pi$.

It is well-known that the mapping F is upper semicontinuous. We call F *injective* if $F_x \cap F_y = \emptyset$ for any pair of distinct points $x, y \in S$. Also, we call F *surjective* if for every point $y \in S$ there is some point $x \in S$ with $y \in F_x$. When we say that F is bijective or a homeomorphism, we implicitly state that F is single-valued.

The union of two segments from x to some point $y \in S$, which make an angle equal to π at y, will be called a *loop at x*.

If σ_1, σ_2 are two segments with precisely one common endpoint a, then $\angle\sigma_1\sigma_2$ denotes the angle between the tangent directions of σ_1 and σ_2 at a. For $a \neq b$, ab means the segment from a to b when that segment is unique or clearly identifiable from the context. $\angle xyz$ means $\angle(xy)(yz)$.

A *geodesic triangle* in a Riemannian manifold or a convex surface is a collection of three segments γ_1, γ_2, γ_3 such that γ_i, γ_{i+1} have a common endpoint a_{i+2} (the indices are taken modulo 3). We shall denote the triangle by $(\gamma_1, \gamma_2, \gamma_3)$ or $a_1a_2a_3$.

Let K denote the sectional curvature of a given Riemannian manifold, and M_H the simply connected 2-dimensional space of constant curvature H.

Put $\rho_x(A) = \inf_{y \in A} \rho_x(y)$ for $A \subset S$. For a point $x \in S$, the *intrinsic radius* at x is $\rho_x(F_x)$. The *radius* of S is defined by $\operatorname{rad} S = \inf_{x \in S} \rho_x(F_x)$, its *diameter* is defined by $\operatorname{diam} S = \sup_{x \in S} \rho_x(F_x)$ and its *injectivity radius* by $\operatorname{inj} S = \inf_{x \in S} \rho_x(C_x)$.

We denote by $D(x, \varepsilon)$ the open disc around x of geodesic radius ε.

Two segments σ_1, σ_2 with a common endpoint *bifurcate* if $\sigma_1 \cap \sigma_2$ includes a non-degenerate arc but none of the two segments.

2 Auxiliary results

The set $\mathscr{S}_2$ of all convex surfaces possessing some point $x \in S$ with disconnected M_x is obviously of second Baire category in $\mathscr{S}$. It was introduced by the second author in [15], where he showed that, in the sense of Baire category, on most $S \in \mathscr{S}_2$ there exist a point x and a Jordan arc in $C(x)$ containing infinitely many points of M_x.

Other properties of F were established in [11], where the following two lemmas are proved.

Lemma 1. *The mapping F is injective on any surface $S \in \mathscr{S}$ without conical points, in particular on any surface of class C^1.*

Lemma 2. *Let $S \in \mathscr{S}$. If F is continuous then it is surjective. If F is surjective then F_x is connected for each $x \in S$.*

The following two results were established in [12].

Lemma 3. *For any surface $S \in \mathscr{S}$ and any point $x \in S$, each component of F_x is a point or an arc.*

Lemma 4. *If $S \in \mathscr{S}$, $x \in S$ and $y \in M_x$, then each arc in T_y of length π contains the tangent direction of a segment from y to x. Thus, if $\lambda T_y > \pi$, then there are at least two segments from x to y, and if S is differentiable at y and there are only two segments from x to y then these have opposite tangent directions at y.*

We shall make use of the next lemma, a proof of which can be found in [15] (see also [13]).

Lemma 5. *Let $S \in \mathscr{S}$, $x \in S$ and $y, z \in C_x$. Let J be the arc joining y to z in C_x. If $u \in J \setminus \{y, z\}$ is a relative minimum of $\rho_x|_J$, then u is the midpoint of a loop Λ at x and, except for the two subarcs of Λ from x to u, no segment connects x to u.*

Moreover, we shall need the following classical relation of Clairaut (see, for example, [4] p. 257).

Lemma 6. *Let S be a surface of revolution. For a variable point x on a geodesic γ of S, denote by r_x the distance from x to the axis of revolution, and by θ_x the angle made at x by γ with the meridian through x. Then $r_x \sin \theta_x$ is constant as x varies on γ.*

We shall also use the following results.

Lemma 7 ([10]). *For an arbitrary point x on a closed convex surface S centrally symmetric about the origin, $F_x = \{-x\}$ if and only if $F_y = \{x\}$ for all $y \in F_x$.*

Lemma 8 ([1]). *On any convex surface, segments do not bifurcate.*

Lemma 9 ([1]). *If* ab, bc *are segments on* $S \in \mathscr{S}$, *and* $x \in ab$ *converges to* b *while* xc *converges to* bc, *then* $\angle axc$ *converges to* $\angle abc$.

The following comparison theorem can be found, for example, in [1].

Lemma 10. *Let* S *be a convex surface,* $(\gamma_1, \gamma_2, \gamma_3) \subset S$ *a geodesic triangle and* $(\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3)$ *a planar triangle with* $\lambda\gamma_i = \lambda\bar{\gamma}_i$. *Then*

$$\angle\bar{\gamma}_i\bar{\gamma}_{i+1} \leqslant \angle\gamma_i\gamma_{i+1} \quad (i = 1, 2, 3 \bmod 3).$$

Toponogov's well-known comparison theorem, reproduced here as Lemma 11, can be found, for example, in [2].

Lemma 11. *Let* M *be a complete manifold with* $K \geqslant H$, *and* $(\gamma_1, \gamma_2, \gamma_3)$ *a geodesic triangle in* M. *If* $H > 0$, *suppose* $\lambda\gamma_i \leqslant \pi/\sqrt{H}$ *for all* i. *Then there exists in* M_H *a geodesic triangle* $(\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3)$ *such that* $\lambda\gamma_i = \lambda\bar{\gamma}_i$ *and the corresponding angles satify* $\angle\bar{\gamma}_i\bar{\gamma}_{i+1} \leqslant \angle\gamma_i\gamma_{i+1}$ $(i = 1, 2, 3)$.

If the inequality $K \leqslant H$ *is assumed, then the inequalities* $\angle\bar{\gamma}_i\bar{\gamma}_{i+1} \geqslant \angle\gamma_i\gamma_{i+1}$ *hold* $(i = 1, 2, 3)$.

We shall also use Pogorelov's rigidity theorem ([6] p. 167):

Lemma 12. *Any two isometric convex surfaces are congruent.*

The following is folklore.

Lemma 13. *The orthogonal projection of any arc outside a compact convex surface onto the surface is not longer than the arc itself.*

3 Surfaces with F a homeomorphism, but not an involution

The classical Sphere Theorem (see [2], [9], or [5]) gives sufficient conditions for a complete, simply connected manifold M with Gauß curvature K to be homeomorphic to the unit sphere: $1/4 < K < 1$. In fact, this 1/4-pinching of the curvature proves sufficient for F to be a homeomorphism.

Theorem 1. *Let* S *be a convex surface of class* C^2 *with Gauß curvature* $K \geqslant 1$. *If* $\operatorname{rad} S > \pi/2$ *then* F *is a homeomorphism.*

Proof. The injectivity of F follows from Lemma 1. The single-valuedness of F and its surjectivity can in fact be found inside the proof of Theorem 3 in [5]. For the readers convenience, we give here a short direct proof.

Suppose there exists $x \in S$ with $\operatorname{card} F_x > 1$. Then choose two points $y, z \in F_x$, an arc J joining y to z in C_x and a relative minimum $u \in J \setminus \{y, z\}$ of $\rho_x|_J$. By Lemma 5, u is the midpoint of a loop Λ at x, so both subarcs of Λ from x to u are segments and make the angle π at u.

Let uy be a segment from u to y. Then one of the two angles determined by Λ and uy is at most $\pi/2$. Consider the triangle $xuy \subset S$ containing that angle, and a triangle $\bar{x}\bar{u}\bar{y} \subset M_1$ isometric to xuy.

Comparing the triangles xuy and $\bar{x}\bar{u}\bar{y}$, by Lemma 11, we have $\angle \bar{x}\bar{u}\bar{y} \leqslant \pi/2$. Since $\lambda\bar{x}\bar{y} = \rho(x, y) \geqslant \operatorname{rad} S > \pi/2$, $\bar{y}$ lies in the open half-sphere of M_1 opposite to $\bar{x}$. Then, the inequality $\lambda\bar{x}\bar{u} \leqslant \lambda\bar{x}\bar{y}$ implies $\angle \bar{x}\bar{u}\bar{y} > \pi/2$, and a contradiction is obtained. Thus, F is single-valued on S and therefore continuous. By Lemma 2, F is also surjective.

Since S is compact and F is bijective and continuous, its inverse is also continuous. □

Remark 1. After rescaling, Theorem 1 says that, if $S \in \mathscr{S}$, $K \geqslant k_0 > 0$ and $\operatorname{rad} S > \pi/(2\sqrt{k_0})$, then F is a homeomorphism.

Corollary. *If $1/4 \leqslant K < 1$ then F is a homeomorphism.*

Proof. By Klingenberg's inequality in [2], p. 98, $K < 1$ implies $\operatorname{inj} S > \pi$, whence $\operatorname{rad} S > \pi$. Thus, by Remark 1 with $k_0 = 1/4$, $K \geqslant 1/4$ implies that F is a homeomorphism. □

Remark 2. The corollary says that, if $S \in \mathscr{S}_2$ has everywhere positive Gauß curvature and $K_{\min}$, $K_{\max}$ denote its minimum and maximum respectively, then $K_{\max} \geqslant 4K_{\min}$. The converse is, however, false, as we can easily see on some surfaces in the class $\mathscr{R}$ defined in [10].

We treat now a special case, which will give the answer to the problem mentioned in the Introduction. We consider surfaces S of class C^2 which are of revolution about the axis $\Omega = oZ$ and symmetric with respect to the origin o of the space. The *equator* Q is the intersection of S with the plane through o orthogonal to Ω. A *meridian* is the intersection of S with a plane including Ω.

Denote by $\mathscr{N}$ the set of all surfaces S satisfying the following conditions.

i) The curvature K is pinched, $1/4 \leqslant K < 1$, with the minimum $1/4$ attained on Q.

ii) The radius of Q is less than $\sqrt{3}$.

iii) The surface S surrounds the sphere Σ of equator Q and $S \cap \Sigma = Q$.

We observe that $\mathscr{N}$ contains the ellipsoids of revolution with semiaxes $\sqrt{2} < a = b < c \leqslant 2$.

Theorem 2. *On any surface in $\mathscr{N}$, the mapping F is a homeomorphism but not an involution.*

Proof. Since the Gauß curvature on S verifies $1/4 \leqslant K < 1$, F is a homeomorphism of S, by the previous corollary. It remains to prove that $F \circ F \neq \mathrm{id}_E$.

Clearly, by symmetry, x and F_x are on the same meridian, for any point $x \in S$. Let $y \in Q$, and let $x \in S \setminus \{y\}$ lie on the same meridian. If x is close enough to y, then there is a unique segment xy joining x to y. We show that, for $\rho(x, y)$ small enough, $F_x \neq \{-x\}$. This will imply, by Lemma 7, that F is not an involution.

First remark that, by Lemma 13, any meridian is longer than the equator (in case of equality, $S = \Sigma$), and Q is a geodesic of S.

Now, assume $F_x = \{-x\}$. Clearly, any segment σ from x to $-x$ intersects Q. The existence of two such intersection points, would contradict Lemma 8. Thus, $\sigma \cap Q$ is a single point. No segment σ (from x to $-x$) can lie on a meridian M, for x close enough to y, because M is longer than Q. By Lemma 4, some segment σ from x to $-x$ makes an angle $\alpha \geqslant \pi/2$ with xy. Let z be the equatorial point of σ.

Case $\alpha = \pi/2$: In this case, by Lemma 6, σ is horizontal at $-x$, and the symmetry of the whole geodesic $\Gamma \supset \sigma$ with respect to the line oz implies that z is the midpoint of σ. Since the curvature of S along Q is $K = 1/4$, in a whole neighbourhood N of Q, $K \leqslant 1/3$. Because σ converges to a half-equator if x tends to y, σ lies in N if xy is short enough.

Therefore, on the sphere $M_{1/3}$ of radius $\sqrt{3}$, the triangle $\bar{x}\bar{y}\bar{z}$ isometric to xyz has its angles at $\bar{x}$ and $\bar{y}$ not smaller than $\pi/2$, by Lemma 11. On $M_{1/3}$ this implies, $\bar{x}$, $\bar{y}$ being close, that $\rho(y, z)$ is at least a quarter of the circumference $2\pi\sqrt{3}$, i.e., $\rho(y, z) \geqslant \pi\sqrt{3}/2$, in contradiction with $\rho(y, z) = \pi a/2 < \pi\sqrt{3}/2$, where a is the radius of Σ.

Case $\alpha > \pi/2$: Now σ has some point $x' \neq x$ as a farthest point from Q. Let y' be the equatorial point closest to x'. By Lemma 6 and by the symmetry of S and of Γ, there is a unique point $x'' \in \sigma$ between x and z at distance $\rho(x, y)$ from Q, and the point $-x$ is either symmetric to x or to x'' with respect to the line oz. In the first case $\rho(z, y') < \pi a/2$. In the second, Γ reaches $-x'$ beyond $-x$, $\rho(-x, -x') = \rho(x'', x') = \rho(x, x')$, and $\rho(z, y') = \pi a/2$.

We obtain the geodesic triangle $x'y'z$, to which we apply the argument from the previous case, and obtain a contradiction. □

4 Surfaces with involutive F

Denote by $\mathscr{H}$ the subset of $\mathscr{S}$ of all convex surfaces which satisfy the conditions in Steinhaus' conjecture mentioned in the Introduction: F is single-valued and an involution.

In [10] the first author constructed a class $\mathscr{R} \subset \mathscr{H}$ of convex surfaces larger than that of all 2-spheres, thus disproving the mentioned conjecture, and asked for a characterization of $\mathscr{H}$.

Here we find an even larger family of examples of surfaces in $\mathscr{H}$, namely the family $\mathscr{I}$ of all surfaces $S \in \mathscr{S}$ of constant intrinsic radius, i.e.,

$$\mathscr{I} = \{S \in \mathscr{S} : \operatorname{rad} S = \operatorname{diam} S\}.$$

The relationship between $\mathscr{R}$, $\mathscr{I}$ and $\mathscr{H}$ is not obvious. It is described by the next theorem.

Theorem 3. *We have $\mathscr{R} \subset \mathscr{I} \subset \mathscr{H}$, where the first inclusion is strict.*

Proof. We show that $\mathcal{I} \subset \mathcal{H}$.

Let $S \in \mathcal{I}$. Notice that $\rho_x(F_x) = \operatorname{diam} S$ for all $x \in S$. Therefore, for any points $x \in S$, $y \in F_x$, we have $x \in F_y$, which implies that F is surjective. By Lemma 2, for every $x \in S$ the set F_x is connected, and by Lemma 3 it must be an arc or a point. Clearly, F is involutive if it is single-valued.

We claim that F_x actually reduces to a point for all $x \in S$. Suppose there is a point $x \in S$ with F_x a nondegenerate arc. By Lemma 1, x is a conical point.

Let z_1, z_2 be the endpoints of F_x. Denote by $\Delta \subset S$ the maximal open connected set not meeting any segment from x to z_1 or z_2, but meeting F_x, and by S_1, S_2 the two components of $S \backslash \overline{\Delta}$ with $z_i \in \operatorname{bd} S_i$ $(i = 1, 2)$. Let α, β be the two angles of $\overline{\Delta}$ at x, and γ_i the angle of $\overline{S}_i$ at x $(i = 1, 2)$. Since $\Delta \cup S_i$ meets no segment from x to z_i, by Lemma 4, $\gamma_i + \alpha + \beta \leqslant \pi$.

For any $v \in F_x \backslash \{z_1, z_2\}$, there is a loop Λ_v at x through v, by Lemma 5. Since limits of segments are also segments, the same is true for $v \in \{z_1, z_2\}$. So, for any $v \in F_x$, denote by σ_v, σ'_v the two segments from x to v forming Λ_v, and by $\delta_i(v)$ the angle of σ_v, σ'_v at x toward S_i. Since $\delta_1(z_1) = \gamma_1$ and $\delta_1(z_2) = \gamma_1 + \alpha + \beta$, there is some point $z \in F_x$ such that

$$\delta_i(z) = \gamma_i + v_i,$$

where $v_i > 0$, $v_1 + v_2 = \alpha + \beta$, and $\gamma_i + v_i \neq \pi/2$ $(i = 1, 2)$. Let $\varphi < \pi/2$ satisfy $\varphi > \delta_i(z)$ if $\delta_i(z) < \pi/2$ and

$$\varphi > \max\{\delta_i(z)/2, \pi - \delta_i(z)\}$$

if $\delta_i(z) > \pi/2$.

Let now $\varepsilon > 0$ be such that the planar triangle with one side of length ε, another side of length $\operatorname{rad} S/2$ and the angle between them φ, is obtuse.

By the semicontinuity of F, if $y \in \sigma_z$ is close enough to z, then $F_y \subset D(x, \varepsilon)$. Let D_1, D_2 be the two components of $D(x, \varepsilon) \backslash \Lambda_z$, and E_1, E_2 the two components of $S \backslash \Lambda_z$ satisfying $D_i \subset E_i$ $(i = 1, 2)$. Let $y^* \in \sigma_z$ have maximal distance to y in $\overline{E}_1$. This maximal distance is less than $\operatorname{diam} S$. Indeed, let y' be the point of σ'_z at distance $\rho(y, z)$ from x. Clearly, $yx \cup xy'$ is not a segment, x being a conical point, so $\rho(y, y') < \operatorname{diam} S$. Also, any other point of Λ_z is at distance less than $\operatorname{diam} S$ from y, whence $F_y \cap \Lambda_z = \emptyset$.

It is easily seen that there is at least one segment μ_1 from y to y^* such that $\angle \mu_1 y^* x \leqslant \pi/2$, and at least one segment μ_2 from y to y^* such that $\angle \mu_2 y^* x \geqslant \pi/2$. Indeed, suppose there is no segment of at least one of the two kinds, for instance of the first kind. Then, for $w \in y^* x$ converging to y^*, any segment from y to w necessarily converges to yy^*. So, by Lemma 9,

$$\angle ywy^* \to \pi - \angle \mu_1 y^* x < \pi/2.$$

Comparing with the planar triangle whose vertex set is isometric to $\{y, w, y^*\}$, we get by Lemma 10 the inequality $\rho(y, w) > \rho(y, y^*)$, which is false.

Now let $y \to z$. Consequently, $y^* \to x$. Since there is no segment from x to z besides σ_z, σ_z', we have $\mu_1 \to \sigma_z$ and $\mu_2 \to \sigma_z'$. Suppose no segment from y to y^* is included in Λ_z. Then, still, for y close enough to z, μ_2 meets F_x, say at z'. Then

$$2\rho(x, z') \leqslant \rho(x, y) + \lambda\mu_2 + \rho(y^*, x) < \lambda\Lambda_z = 2 \operatorname{rad} S,$$

which is false. Hence, $\mu_2 = yz \cup zy^*$.

There are two possibilities for F_y to meet D_1. Let $u \in F_y \cap D_1$.

Case 1. u lies in the digon of sides μ_1, μ_2.

Case 2. u lies in the triangle (μ_1, y^*x, xy).

In the following discussion we assume for both cases that $\delta_1(z) > \pi/2$. (The contrary situation can be treated analogously and is simpler.)

In Case 1, some triangle yy^*u has its angle at y^* not larger than $\angle zy^*\mu_1/2$ and therefore less than φ for y close enough to z, because $\angle zy^*\mu_1 \to \delta_1(z)$ by Lemma 9. Looking at the Euclidean triangle with vertex set isometric to $\{y, y^*, u\}$, we get $\rho(y, u) < \rho(y, y^*)$ by Lemma 10; this contradicts $u \in F_y$.

In Case 2, the triangle yy^*x either has its angle at y^* less than φ, or its angle at x less than φ, for y close enough to z. In the first situation, the triangle yy^*u has its angle at y^* less than φ, hence, as before, $\rho(y, u) < \rho(y, y^*)$, and a contradiction is obtained. In the second situation, the triangle yux has its angle at x less than φ, whence $\rho(y, u) < \rho(y, x)$, again a contradiction.

Thus F_y does not meet D_1. Analogously, it does not meet D_2. Since we showed that F_y does not meet Λ_z either, we found that $F_y \cap D(x, \varepsilon) = \emptyset$, and a final contradiction is obtained. The proof is almost finished.

From Theorem 9 and Remark 10 of [10], it follows that $\mathscr{R} \subset \mathscr{I}$. J. Rouyer [8] showed that this inclusion is strict. He proved that the boundary of a half-ball belongs to $\mathscr{I} \setminus \mathscr{R}$. □

For a given convex surface S, endow the space $\mathscr{P}(S)$ of all compact subsets of S with the induced Pompeiu–Hausdorff metric $\mathscr{H}_\rho$.

Theorem 4. *The surface $S \in \mathscr{S}$ is a centrally symmetric surface in $\mathscr{H}$ if and only if the associated mapping F is an isometry.*

Proof. Let $S \in \mathscr{H}$ be centrally symmetric about the origin. By Lemma 7, $F_x = \{-x\}$ for all $x \in S$, hence F is the restriction to S of the symmetry with respect to the origin, and therefore an isometry of S.

Conversely, if there is a point $x \in S$ with $\operatorname{card} F_x > 1$, then we can find a sequence x_n tending to x such that $\operatorname{card} F_{x_n} = 1$, and F_{x_n} converges to a point $z \in F_x$ (see Theorem 5 in [14]). In this case, we have $\rho(x_n, x) \to 0$ and $\mathscr{H}_\rho(F_{x_n}, F_x) \to \mathscr{H}_\rho(\{z\}, F_x) > 0$. Thus, F is an isometry between the metric spaces (S, ρ) and $(\mathscr{P}(S), \mathscr{H}_\rho)$ if and only if it is single-valued and an isometry of (S, ρ). So, let F be such a mapping.

From $\rho(x, F_x) = \rho(F_x, F_{F_x})$, it follows that $F_{F_x} = x$, hence $S \in \mathscr{H}$. By Pogorelov's rigidity theorem (Lemma 12), the isometric convex surfaces S and $F(S)$ are congru-

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ent via an extension f of the isometry F to the whole space. Since f leaves S invariant and has no fixed points on S, it must be the symmetry with respect to the midpoint of some line-segment joining a point x to its (unique) farthest point. □

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