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ON THE NUMBER OF SHORTEST PATHS BETWEEN POINTS ON MANIFOLDS

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Abstract. A result of P. Horja about n-dimensional compact Riemannian manifolds of non-positive curvature says that, from any point a of the manifold to any relative maximum b of the distance function from a, there are at least $n+1$ shortest paths. Without the curvature condition involved above, we are only sure about the existence of 2 shortest paths. In this note we present weeker curvature conditions yielding Horja's conclusion.
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H. Steinhaus [6] proved that between a point a of a closed surface $S \subset \mathbb{R}^3$ and any farthest point $b \in S$ from a (in the intrinsic metric) there are at least 2 segments (i. e. shortest paths); and in the case of precisely 2 segments, they have opposite directions at y. In Steinhaus' approach S was considered of class C^2. The corresponding result was established for arbitrary convex surfaces and further strengthened in [7].

On the other hand, P. Horja [4] showed that, on n-dimensional compact Riemannian manifolds of non-positive curvature, from any point a to any relative maximum b of the distance function from a there are at least $n+1$ segments.

Here we find weaker sufficient curvature conditions for the existence of $n+1$ segments from a to b. We implicitely contribute to the study of the set of all farthest points from a, which Steinhaus had asked for (see [3]).

Let $\mathcal{M}$ be an n-dimensional topological manifold possessing an intrinsic metric ρ ($n \geq 2$) and consider a point $a \in \mathcal{M}$.

The minimal number 2 of segments found by Steinhaus is not related to the dimension of S. It remains 2 even in the case of higher-dimensional surfaces. Instead, it is related to the property of a relative maximum of $\rho_a = \rho(a, \cdot)$ of being a critical point.

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A point $p \in \mathcal{M}$ is called *critical with respect to* ρ_a, or simply *critical*, if for any direction τ at p there is a geodesic from a to p with direction τ' at p, such that $\angle(\tau, \tau') \leq \pi/2$ (see, for example, [2]).

Let $p \in \mathcal{M}$ be called *strictly critical* if for any direction τ at p there is a geodesic from a to p with direction τ' at p, such that $\angle(\tau, \tau') < \pi/2$.

In the cases we are going to use these definitions, every geodesic from a to p has a direction at p.

Suppose the space of directions at p is isometric to S^{n-1}. It is now obvious that the point p can be critical if only 2 segments from a to p exist, under the condition that their directions in p are opposite. An ellipsoid in $\mathbb{R}^{n+1}$ with all axis-lengths distinct and with a, p realizing its diameter provides a concrete example. On the contrary, if p is strictly critical, then there always exist at least $n+1$ segments from a to p.

We are going to use the concept of an *Alexandrov n-space*; it will always refer to a space of curvature bounded above (by some real number K). More precisely, it will be a domain R_K in the terminology of Berestovskij and Nikolaev [1] (see p. 185), and an n-dimensional topological manifold ($n \geq 2$), with or without boundary.

Theorems 1 and 2 below and the Corollaries refer to a compact connected metric space $\mathcal{X}$ endowed with the intrinsic metric ρ, in which a is an arbitrary point, b is a relative maximum of ρ_a, and $\mathcal{S}$ is the collection of all segments from a to b.

Theorem 1. *Let $\mathcal{S}' \subset \mathcal{S}$ consist of m segments, where $m \leq n$. Suppose that for each segment $\sigma \in \mathcal{S}'$ there exists an Alexandrov n-space $A_\sigma \subset \mathcal{X}$ with the intrinsic metric induced by ρ and with curvature bounded above by $\kappa < \pi^2/4\rho(a,b)^2$, such that $\sigma \subset A_\sigma$ and the space of directions at b is isometric to S^{n-1}. Then $\mathcal{S}$ contains at least $m+1$ segments.*

Proof. Let j be the isometry between the space of directions at b and S^{n-1}. For each direction τ at b there is a geodesic with direction τ at b (see Propositions 8.3 and 8.2 in [1], pp. 195, 194). Suppose $\mathcal{S}' = \{\sigma_1, \sigma_2, ..., \sigma_m\}$. If the direction of σ_i at b is τ_i ($i = 1, ..., m$), then there is some $\tau \in S^{n-1}$ satisfying $\angle(\tau, j(\tau_i)) \geq \pi/2$ for all i.

Let σ be a geodesic with tangent direction $j^{-1}(\tau)$ at b. Let N be a neighbourhood of b such that $\rho(a,b) \geq \rho(a,u)$ for each $u \in N$, and σ is length minimizing between b and u for each $u \in \sigma \cap N$. Take a point $c \in \sigma \cap N$.

Let $\sigma(c)$ be a segment from a to c. When c converges to b, there is at least one limit segment σ' of $\sigma(c)$, joining a to b. (See, for example [5], p. 143.) If this is not in $\mathcal{S}'$, we are ready. Suppose now σ' coincides with one of the segments in $\mathcal{S}'$, say with σ_1. Then the induced intrinsic distance $\rho^*(a,c)$ in A_{σ_1} coincides with $\rho(a,c)$ for all points c in some neighbourhood of b contained in N. Of course, $\rho^*(b,c) = \rho(b,c)$ too.

Comparing the triangle abc in A_{σ_1} with the isometric triangle $a^*b^*c^*$ on the 2-sphere Σ of constant curvature κ,

$$\angle a^*b^*c^* \geq \angle abc \geq \pi/2.$$

Due to the condition $\kappa < \pi^2/4\rho(a,b)^2$, the radius r of Σ must satisfy $r > 2\rho(a,b)/\pi$. Hence a^*b^*, the distance between a^* and b^* on Σ, is less than 1/4 of the length of a great circle, and a^*c^* is not larger than a^*b^*, which implies that $\angle a^*b^*c^* < \pi/2$, and a contradiction is obtained.

Theorem 2. *Suppose that for each segment $\sigma \in \mathcal{S}$ there exists an Alexandrov n-space $A_\sigma \subset \mathcal{X}$ with the intrinsic metric induced by ρ and with curvature bounded above by $\kappa < \pi^2/4\rho(a,b)^2$, such that $\sigma \subset A_\sigma$ and the space of directions at b is isometric to S^{n-1}. Then $\mathcal{S}$ contains at least $n+1$ segments and b is a strictly critical point.*

Proof. The first assertion follows not only from the second, but also from a repeated application of Theorem 1.

To prove the second assertion, suppose that b is not a strictly critical point and let $\tau \in S^{n-1}$ verify $\angle(\tau, j(\tau')) \geq \pi/2$ for the tangent direction τ' at b of every segment $\sigma' \in \mathcal{S}$. Choose σ, N, c and $\sigma(c)$ as in the proof of Theorem 1. As c converges to b, any limit segment of $\sigma(c)$ belongs to $\mathcal{S}$. The same argument as in the proof of Theorem 1 leads now again to a contradiction.

We mention several consequences of Theorem 2.

Corollary 1. *Suppose $\mathcal{X}$ is a Riemannian manifold with sectional curvature at most $\kappa < \pi^2/4\rho(a,b)^2$ on an open set including all segments from $\mathcal{S}$. Then b is a strictly critical point.*

In case the curvature condition is satisfied everywhere, we obtain the next corollary.

Corollary 2. *Suppose $\mathcal{X}$ is a Riemannian manifold with sectional curvature at most $\kappa < \pi^2/4\rho(a,b)^2$. Then b is a strictly critical point.*

We can strengthen the curvature condition not to depend on a and b.

Corollary 3. *Suppose $\mathcal{X}$ is a compact Riemannian manifold with diameter Δ and sectional curvature at most $\kappa < \pi^2/4\Delta^2$. Then b is a strictly critical point.*

Corollary 3 and – a fortiori – Corollaries 1 and 2 are direct strengthenings of Horja's mentioned result [4].

The next corollary is another direct consequence of Theorem 2.

Corollary 4. *Suppose $\mathcal{X}$ is the boundary of a convex polyhedron in the n-dimensional sphere of radius larger than $2\rho(a,b)/\pi$ and b is not a vertex. Then b is a strictly critical point.*

To use Theorem 2, take the convex hull Ξ of the union of a point x in the given n-sphere Σ and a small ball in Σ around a point at distance $\rho(a,b)$ from x; then use isometric copies of Ξ as Alexandrov n-spaces, with the point x corresponding to a.

As a limit case we have the following.

Corollary 5. *Suppose $\mathcal{X}$ is the boundary of a convex polyhedron in $\mathbb{R}^{n+1}$ and b is not a vertex. Then b is a strictly critical point.*

As already noticed, the (common) conclusion of Corollaries 1 - 5 yields the existence of $n+1$ segments from a to b.

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