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TETRAHEDRA PASSING THROUGH A CIRCULAR OR SQUARE HOLE

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Abstract. Suppose a plane in $\mathbb{R}^3$ has a hole. In this paper we determine the smallest circular and the smallest square hole through which a regular tetrahedron of given size can pass.

MSC 2000: 52C99, 52A99

In [1] two of us looked for the shape of a convex hole H of diameter and width as small as possible such that the regular tetrahedron T of edge length 1 can pass through it. They found such a hole with diameter $\sqrt{3}/2$, the width of a face of T, and with width $\sqrt{2}/2$, the width of T.

In this paper we look for holes of given shape and as small as possible, allowing the moving tetrahedron T to pass through it. We successively treat the cases when the hole is a disk and a square. Eighty-five years ago, K. Zindler [3] already considered convex bodies moving through a circular hole. For a special polytope, an affine cube, he found surprisingly small holes allowing it to pass through. We shall see here that this is so for the tetrahedron T too.

Let p_1, p_2, p_3, p_4 be the (variable) vertices of the regular tetrahedron T of edge-length 1 and of variable position in $\mathbb{R}^3$, and $P \subset \mathbb{R}^3$ the plane which will contain the hole.

Theorem A. *Assume that the hole $H \subset P$ is a disk. The smallest diameter of H such that T can pass through H is*

$$\frac{t_0^2 - t_0 + 1}{\sqrt{\frac{3}{4}t_0^2 - t_0 + 1}} = 0.8957...,$$

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where

$$t_0 = \frac{2 + \sqrt[3]{\sqrt{43} - 4} - \sqrt[3]{\sqrt{43} + 4}}{3}.$$

Theorem B. *Assume that the hole $H \subset P$ is a square. The smallest such hole allowing T to pass through it has diagonal length* 1.

For the proof of Theorem A we need the following lemma.

Lemma 1. *Let $q_i(s) \in p_1p_i$ be at distance s from p_1 $(i = 2, 4)$. The diameter of the circle circumscribed to the triangle $p_3q_2(s)q_4(t)$ is minimal precisely when $s = t = t_0$.*

Proof. First we claim that if the radius $r(s,t)$ of the circle circumscribed to the triangle $p_3q_2(s)q_4(t)$ is minimal, then $s = t$.

Consider indeed $s \neq t$. By symmetry, the circles C', C'' circumscribed to $p_3q_2(s)q_4(t)$ and $p_3q_2(t)q_4(s)$ are congruent, and meet at p_3 and at another point q on the plane through p_1, p_3, and $(p_2 + p_4)/2$.

The circles C', C'' lie on the sphere determined by p_3, $q_2(s)$, $q_2(t)$, $q_4(s)$, $q_4(t)$. The plane P' through q and p_3 parallel to p_2p_4 intersects the sphere along a circle C. This circle meets the plane determined by p_1, p_2, p_4 at two points which belong to the arcs $q_2(s)q_2(t)$ and $q_4(s)q_4(t)$ of the circle through $q_2(s)$, $q_2(t)$, $q_4(s)$, $q_4(t)$.

Let $\{q_i'\} = p_1p_i \cap P'$ $(i = 2, 4)$. The circle circumscribed to $p_3q_2'q_4'$ is obviously smaller than C. The circle C is smaller than or congruent to the circles C', C'', with congruence if qp_3 is a diameter of the sphere. Thus, $r(s,t)$ is not minimal, and this proves our claim.

Now, direct calculation yields

$$\|p_3 - q_i(t)\| = \sqrt{t^2 - t + 1} \qquad (i = 2, 4)$$

and, for the height h^* of $p_3q_2(t)q_4(t)$ at p_3,

$$h^* = \sqrt{\frac{3}{4}t^2 - t + 1}.$$

Hence

$$r(t,t) = \frac{\|p_3 - q_i\|^2}{2h^*} = \frac{t^2 - t + 1}{2\sqrt{\frac{3}{4}t^2 - t + 1}}.$$

The derivative of $f(t) = r(t,t)$ is

$$f'(t) = \frac{3t^3 - 6t^2 + 7t - 2}{8\left(\frac{3}{4}t^2 - t + 1\right)\sqrt{\frac{3}{4}t^2 - t + 1}}.$$

It turns out that the equation $3t^3 - 6t^2 + 7t - 2 = 0$ has the only solution t_0 in the interval $[0,1]$. Hence $f|_{[0,1]}$ takes its minimal value when $t = t_0$. This and the above claim imply that $r|_{[0,1]\times[0,1]}$ has its unique minimum at (t_0, t_0).

Proof of Theorem A. ¿From Lemma 1 it is clear that T cannot pass through a hole H with radius less than $f(t_0)$. Let H_0 be the disk with radius $f(t_0)$.

Now we show how T passes through H_0.

Let $q_{ij} \in p_i p_j$ be at distance t_0 from p_i. We easily see that the circles $q_{12}q_{14}q_{32}q_{34}$ and $q_{21}q_{23}q_{41}q_{43}$ are congruent and lie in parallel planes. Their convex hull is a cylinder including all intersections of intermediate planes with T.

So the tetrahedron T passes through H starting with p_1, then puts its points q_{12}, q_{14} on the boundary of H, rotates around $q_{12}q_{14}$ - during this rotation the vertex p_3 passes, by Lemma 1, through H - until $q_{32}, q_{34} \in P$. Then a translation brings T in a position with the circle $q_{12}q_{14}q_{32}q_{34}$ concentric with H. Another (orthogonal) translation makes the circle $q_{21}q_{23}q_{41}q_{43}$ concentric with H. Then the moves are as before but in inverse order, and T escapes from H into the other half-space.

The Theorem in [2] shows that most convex bodies can be held by a circle. From Theorem A we now know more about the size of such a circle in case the convex body is a regular tetrahedron.

Corollary. *The regular tetrahedron T of edge length 1 can be held precisely by the circles of diameter d satisfying*

$$0.7071... = \frac{\sqrt{2}}{2} \le d < \frac{t_0^2 - t_0 + 1}{\sqrt{\frac{3}{4}t_0^2 - t_0 + 1}} = 0.8957...$$

To prove Theorem B we need the following lemmas.

Since

$$\frac{3}{2}t - 1 + \sqrt{\frac{3}{4}t^2 - t + 1} \ge 0$$

is equivalent to $t(2-\frac{3}{2}t) \geq 0$, which is true in $[0,1]$, we have there $f'(t) \geq 0$ and $f(0) = 1$, whence $f(t) \geq 1$ for all t.

So Lemma 2 is proven.

Lemma 3. *With the same notation, the diameter of the smallest square including the triangle $p_3q_2(s)q_4(t)$ is at least 1.*

Proof. The smallest square $\alpha\beta\gamma\delta$ including the triangle $T(s,t)$ has (or can be translated to have) a common vertex with $T(s,t)$. If $\alpha = p_3$, the conclusion follows from Lemma 2.

If $\alpha = q_2(s)$, then $\|\alpha - \beta\| \geq h' \geq h''$, where h' is the height of $p_3q_2(s)q_4(t)$ at p_3 and h'' is the height of T. So $\|\alpha - \gamma\| \geq h''\sqrt{2} = \frac{2}{\sqrt{3}} > 1$.

The third possibility $\alpha = q_4(t)$ is analogous. Hence, in any case, the diameter of $\alpha\beta\gamma\delta$ is at least 1.

Proof of Theorem B. Let H be the square hole with diameter 1.

Since the projection of T on a plane parallel to p_1p_4 and p_2p_3 is a square congruent to H, it is clear that T passes through H.

Now suppose that T passes through a square hole.

If T moves and arrives at the hole with two vertices simultaneously, then the diameter of the hole cannot be less than 1. If one vertex of T goes through the hole first, then – at the moment when the second vertex reaches the hole – we are in the situation of Lemma 3.

This ends the proof.

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