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Acute triangulations of the regular dodecahedral surface

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Abstract

In this paper we consider geodesic triangulations of the surface of the regular dodecahedron. We are especially interested in triangulations with angles not larger than $\pi/2$, with as few triangles as possible. The obvious triangulation obtained by taking the centres of all faces consists of 20 acute triangles.

We show that there exists a geodesic triangulation with only 10 non-obtuse triangles, and that this is best possible.

We also prove the existence of a geodesic triangulation with 14 acute triangles, and the non-existence of such triangulations with less than 12 triangles.

1. Introduction

The notion of a *triangulation* is well known in algebraic topology. In dimension two it means a collection of triangles covering the space, such that the intersection of any two triangles is either empty or consists of a vertex or of an edge. We are interested only in triangulations all the members of which are *geodesic triangles*, i.e., all edges must be shortest paths. This is motivated by the geometric significance of *geodesic triangulations*, i.e., those triangulations using geodesic triangles only. Colin de Verdière [5] shows how to change a triangulation of a compact surface of nonpositive curvature into a geodesic triangulation. The planar case was previously treated by Fary [7] and Tutte [22]. From now on, *triangulation* will always mean a geodesic one.

In rather general two-dimensional spaces, like Alexandrov surfaces, two geodesics starting at the same point determine a well defined angle. Our interest will be focused on triangulations

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which are *non-obtuse* or *acute*, which means that the angles of all geodesic triangles are not larger than or, respectively, smaller than $\pi/2$.

The discussion of non-obtuse and acute triangulations has one of its origins in a problem of Stover reported in 1960 by Gardner in his Mathematical Games section of the Scientific American (see [8,9]). There the question was raised of whether a triangle with one obtuse angle can be cut into smaller triangles, all of them acute.

Independently, in the same year, Burago and Zalgaller [2] deeply investigated acute triangulations of polygonal complexes. Accidentally, a solution to Stover's problem appears in [2]!

Another, even earlier, interest in non-obtuse triangulations stems from the discretization of partial differential equations [17].

In 1980, Cassidy and Lord [3] considered acute triangulations for the surface of a square. Maehara recently investigated acute triangulations of quadrilaterals [18] and other polygons [19], obtaining deeper results. His results on polygons were strengthened by Yuan [23].

Acute triangulations with triangles which are close to equilateral were considered by Gerver [10] and, on Riemannian surfaces, by Colin de Verdière and Marin [6].

On the other hand, Baker, Grosser and Rafferty [1] considered non-obtuse triangulations.

Motivated by the proof of the discrete maximum principle, in 1973 Ciarlet and Raviart [4], Strang and Fix [21], and later Santos [20], were also led to non-obtuse triangulations. Extensions to three dimensions were considered by Křížek and Qun [14], Korotov and Křížek [15] and Korotov et al. [16].

A short survey on acute triangulations is the paper [25].

We started together with Hangan in [11] the investigation of acute triangulations of all Platonic surfaces, which are the surfaces of the five well-known Platonic solids. But besides the trivial cases of the regular tetrahedron and octahedron, only the cube was completely treated. This study was continued for the case of the regular icosahedron by Itoh [12].

Recently, we succeeded in completely solving the problem of finding minimal acute triangulations in the icosahedral case [13]: there is such a triangulation with 12 triangles and there is no such triangulation with fewer triangles.

Here we consider triangulations of the surface of the regular dodecahedron, completely settle the non-obtuse case and find a surprisingly small acute triangulation (with 14 triangles only). The question of whether a triangulation with 12 acute triangles does or does not exist remains open.

It seems that the triangulations of the surface of the regular dodecahedron presented here are (besides those of the cube and of the icosahedron, as treated in [11,12]) the first known non-trivial examples of acute and non-obtuse triangulations of polyhedral surfaces, and we hope that their constructions will possibly give insight into how to obtain such triangulations for other polyhedral surfaces as well. We should mention here the pioneering paper [24] about doubly covered triangles.

Also, we regard our work as a step towards a solution to the following problem first raised in [11]. We consider this problem very natural, and far from trivial.

Problem 1. Does it exist a number N such that every compact convex surface in $\mathbb{R}^3$ admits an acute triangulation with at most N triangles?

Of course, Problem 1 can be extended (or restricted) to other families of surfaces (such as Riemannian), with or without boundary. Even more generally, families $\mathcal{F}$ of two-dimensional

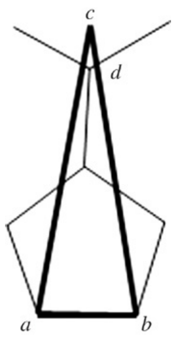

Fig. 1.

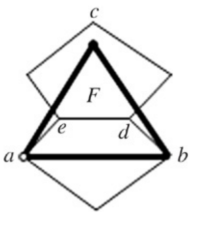

Fig. 2.

triangulable compact topological spaces may be considered. In particular, $\mathcal{F}$ can consist of two-dimensional compact Alexandrov spaces with a common lower (or upper) bound for the curvature. Even the very particular family of all tetrahedral surfaces seems to be quite interesting.

We formulate this as a "meta"-problem.

Problem 2. For interesting families $\mathcal{F}$, investigate the existence in the members of $\mathcal{F}$ of

(a) non-obtuse triangulations,

(b) acute triangulations.

Find a constant N as in Problem 1, if it exists.

2. Non-obtuse triangulations

Let $\mathcal{T}$ be a (geodesic) triangulation of the surface $\mathcal{D}$ of the regular dodecahedron.

It will be convenient to use, besides the intrinsic metric of $\mathcal{D}$, the graph-theoretic distance between vertices of $\mathcal{D}$ regarded as a graph; we call this the *g-distance* between those vertices. Thus, the g-distance between two vertices of $\mathcal{D}$ is the number of edges of the shortest path between those vertices in the 1-skeleton of $\mathcal{D}$.

Triangles *abc* having two vertices at vertices of $\mathcal{D}$ and containing two vertices of $\mathcal{D}$ in their interior are of four types, as described in the following lemma.

Lemma 1. *If a and b are vertices of $\mathcal{D}$ and the triangle abc contains precisely two vertices of $\mathcal{D}$ in its interior then abc is of one of the four types depicted in* Figs. 1–4.

Proof. Type I. a, b are vertices of $\mathcal{D}$ at g-distance 1, and c is behind d but close to d, see Fig. 1.

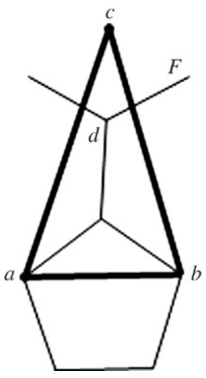

Fig. 3.

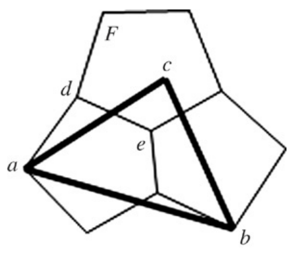

Fig. 4.

This is indeed the only possibility if a, b are vertices of $\mathcal{D}$ at g-distance 1, and the region in which c may lie is easily determined by taking the geodesics through a and b orthogonal to ab.

Type II. a, b are vertices of $\mathcal{D}$ at g-distance 2, and c is in F, behind the centre of F, see Fig. 2.

Like for type I, the geodesics starting at a and b orthogonal to ab in the rough direction of F intersect inside F and determine together with the bisectors of the angles of F at d and e a quadrilateral region in which c can lie.

Type III. a, b are vertices of $\mathcal{D}$ at g-distance 2, and c is behind d, see Fig. 3.

Going in the other direction than in Case II, the geodesics from a and b orthogonal to ab are precisely the bisectors of the edges of F incident to d, meet at the centre of F, and determine in F a quadrilateral where c can be.

Type IV. a, b are vertices of $\mathcal{D}$ at g-distance 3, and c is in the face F, see Fig. 4.

The geodesic G starting at a orthogonal to ab and in direction of F goes very close to ad through the face which contains ad but not e. Suppose ac is close to G. In order for abc to contain just two vertices of $\mathcal{D}$, $\angle ced$ must be at most $\pi/10$. The angle between G and ad at a is less than $3\pi/10$. Therefore $\angle bca > 6\pi/10$, which is not permitted.

Since the geodesic at b orthogonal to ab meets the edge of F at e different from de, there is no possibility other than that stated. It is an elementary exercise to indeed see the existence of a small region possible for c in F.

Suppose now that a and b are at g-distance 4. In this case they are placed like in Fig. 5; the geodesic through b orthogonal to ab arrives at d and is orthogonal to de, where e is the centre of F. So $\angle adb > \pi/2$, which yields $\angle acb > \pi/2$.

Thus, we are only left with the four cases mentioned in the statement.

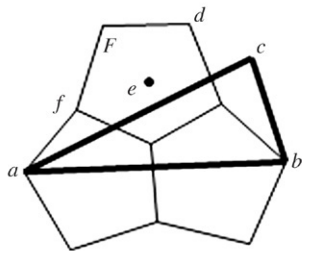

Fig. 5.

Remark 1. By Lemma 1, in no case can the vertex c of the triangle abc of the kind considered in Lemma 1 be at a vertex of $\mathcal{D}$.

Lemma 2. *Let $\mathcal{T}$ contain two neighbouring triangles abc and $a'bc$ with a, a' and b at vertices of $\mathcal{D}$, such that each triangle has two vertices of $\mathcal{D}$ in its interior.*

If $\mathcal{T}$ is non-obtuse then either

(a) *both triangles are of type III,*

or

(b) *one of the two triangles is of type III and the other of type IV.*

If $\mathcal{T}$ is acute then case (a) is excluded.

Proof. Suppose $\mathcal{T}$ is non-obtuse. It is easily seen that, due to the position of c, the only plausible cases for the pair $(abc, a'bc)$ are

1° abc of type II and $a'bc$ of type IV (or vice versa),

2° abc and $a'bc$ of type III,

3° abc of type III and $a'bc$ of type IV (or vice versa),

4° abc and $a'bc$ of type IV.

However, supposing case 4° happens, the sum of the angles at c would be larger than π, which is impossible.

It remains to rule out the case 1°.

Fig. 6 shows the face $mprqn$ containing the vertex c of $\mathcal{T}$ and parts of three neighbouring faces containing at vertices the vertices a, b, a' of $\mathcal{T}$, these faces being unfolded to lie in the plane of $mprqn$.

There are certain restrictions on the position of c. Since $\angle bac \leq \pi/2$, we must have $\angle mac \leq \pi/10$. Since n must remain outside $a'bc$, m and c are separated by the line λ through n orthogonal to pr, or $c \in \lambda$. Let t, u, v be the orthogonal projections of m, b, p on λ and let s be the point of λ (the one closer to v) satisfying $\angle mas = \pi/10$. Obviously, $\angle a'sb \leq \angle a'cb \leq \pi/2$.

We take the edge-length of $\mathcal{D}$ to be 1. Then $mt = \cos\frac{\pi}{5}$ and

$$ts = \left(1 + \cos\frac{\pi}{5}\right)\tan\frac{\pi}{10} = 2\sin\frac{\pi}{10}\cos\frac{\pi}{10} = \sin\frac{\pi}{5}.$$

Also, $nt = \sin\frac{\pi}{5}$ and $un = \cos\frac{\pi}{10}$. Hence

$$us = 2\sin\frac{\pi}{5} + \cos\frac{\pi}{10}.$$

On the other hand, $sv = \cos\frac{\pi}{10} - \sin\frac{\pi}{5}$, while $va' = \frac{1}{2} + 2\cos\frac{\pi}{5}$.

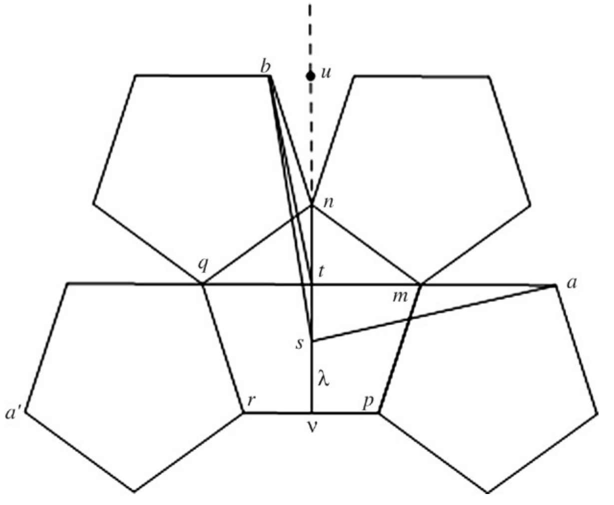

Fig. 6.

Since $\angle a'sb \leq \pi/2$, we must have $\angle usb + \angle vsa' \geq \pi/2$, i.e., $\angle usb \geq \angle va's$ or, equivalently,

$$va' \cdot ub \geq us \cdot sv.$$

This amounts to

$$\left(\frac{1}{2} + 2\cos\frac{\pi}{5}\right)\sin\frac{\pi}{10} \geq \left(2\sin\frac{\pi}{5} + \cos\frac{\pi}{10}\right)\left(\cos\frac{\pi}{10} - \sin\frac{\pi}{5}\right),$$

which can be rewritten as

$$\left(4\cos^2\frac{\pi}{10} - \frac{3}{2}\right)\sin\frac{\pi}{10} \geq \cos^2\frac{\pi}{10}\left(4\sin\frac{\pi}{10} + 1\right)\left(1 - 2\sin\frac{\pi}{10}\right)$$

or even more simply as

$$-\frac{3}{2}\sin\frac{\pi}{10} \geq \cos^2\frac{\pi}{10} - 8\cos^2\frac{\pi}{10}\sin^2\frac{\pi}{10} - 2\cos^2\frac{\pi}{10}\sin\frac{\pi}{10}.$$

Putting here $\sin\frac{\pi}{10} = (\sqrt{5}-1)/4$ and $\cos^2\frac{\pi}{10} = (5+\sqrt{5})/8$, we get successively

$$-\frac{3}{2}\cdot\frac{\sqrt{5}-1}{4} \geq \frac{5+\sqrt{5}}{8} - (5+\sqrt{5})\left(\frac{\sqrt{5}-1}{4}\right)^2 - 2\cdot\frac{\sqrt{5}+5}{8}\cdot\frac{\sqrt{5}-1}{4},$$

$$-3\sqrt{5} + 3 \geq 5 + \sqrt{5} - \frac{\sqrt{5}\cdot 4\cdot(\sqrt{5}-1)}{2} - \frac{1}{2}\cdot\sqrt{5}\cdot 4,$$

$$8 \geq 4\sqrt{5},$$

and this is simply false.

Evidently, case 2° can occur only if $\angle abc = \angle a'bc = \pi/2$. □

Remark 2. If $v_1v_2v_6$ is of type III and $v_2v_3v_6$ of type IV, then Fig. 7 shows the region (bounded by two line segments and a circular arc) where v_6 must be. With those sides of $\mathcal{D}$ which are

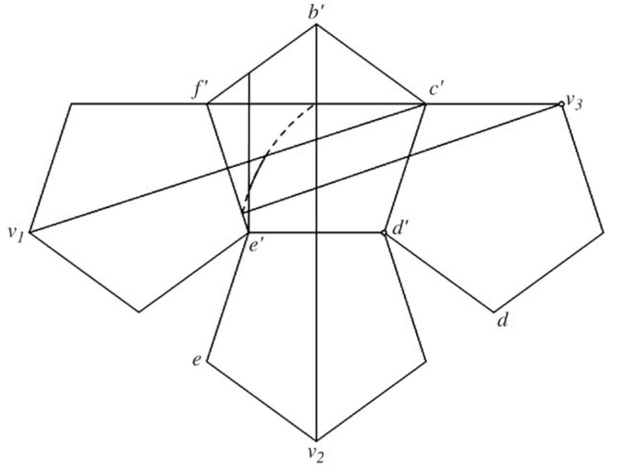

Fig. 7.

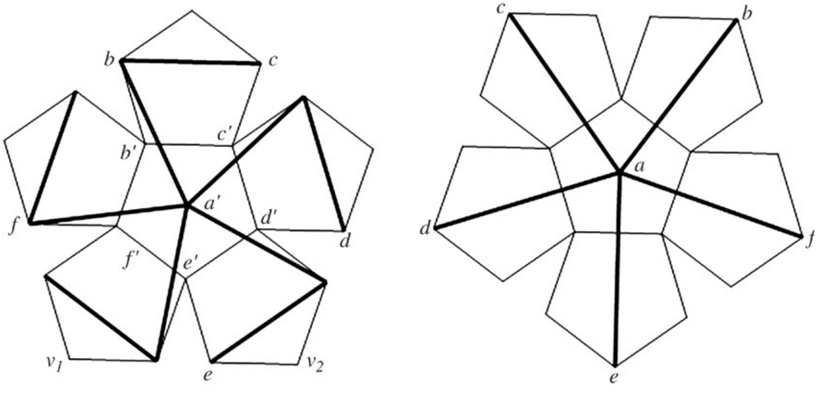

Fig. 8.

neighbours of $b'c'd'e'f'$ rotated to the plane of $b'c'd'e'f'$, the region is determined by the circle of diameter v_2v_3, the line v_1c' and the bisector of the angle $b'e'f'$.

Theorem 1. *The dodecahedron admits a non-obtuse triangulation with 10 triangles, and there are no non-obtuse triangulations with fewer triangles.*

Proof. Fig. 8 describes the surface $\mathcal{D}$ (on the left hand side the upper half of six pentagons, on the right hand side the lower half of six pentagons).

Take the center of the right hand central pentagon F of Fig. 8 and denote it by a. Let a' be the antipodal point of a (i.e., the center of the left hand central pentagon of Fig. 8). Denote the vertices of the five neighbouring pentagons of F which are opposite to the common side with F by b, c, d, e, f as in Fig. 7. Draw the shortest paths from a (resp. a') to b, c, d, e, f and draw the shortest paths bc, cd, de, ef, fb. Note that there are two shortest paths from a' to b

(resp. c, d, e, f); we choose one of them, as in Fig. 7. We get 10 non-obtuse triangles

$$abc, acd, ade, aef, afb, a'bc, a'cd, a'de, a'ef, a'fb.$$

Indeed, all of them are isosceles, the first five are isometric to each other, and the same is true of the last five too. First, we will check the triangle abc. The angle cab is equal to $\frac{2\pi}{5} < \frac{\pi}{2}$. The other angles abc, bca are equal to $\frac{\pi}{2}$. Next we will check the triangle $a'bc$. The angle $ca'b$ is also equal to $\frac{2\pi}{5} < \frac{\pi}{2}$. The angle $a'bc$ is less than $\frac{2\pi}{5}$. The last angle bca' equals $\frac{2\pi}{5} + \angle a'cc'$ ($b'c'd'e'f'$ is the pentagon containing a'), see Fig. 7. Let a'' denote the mid-point of the arc $c'd'$ of the circle circumscribed to $cc'd'd$. Obviously, $\angle a'cc' < \angle a''cc' = \pi/10$, whence $\angle bca' < \pi/2$.

Thus we have obtained a non-obtuse triangulation with 10 triangles of the regular dodecahedral surface.

Now, let us show that there is no non-obtuse triangulation of $\mathcal{D}$ with at most 8 triangles.

Assume there exits such a triangulation $\mathcal{T}$. The degree of any vertex of $\mathcal{T}$ is at least 4, because the total tangent angle at any point of the surface is at least $9\pi/5 > 3\pi/2$. This already excludes the possibility of K_4 as a non-obtuse triangulation. If $\mathcal{T}$ has 6 triangles, it must have 9 edges and, by Euler's formula, 5 vertices. The degree at every vertex being at least 4, there must be at least 10 edges and a contradiction is found.

Similarly, if there are 8 triangles in $\mathcal{T}$, the number of edges is 12 and that of vertices 6. The degree at each vertex must be 4, otherwise a contradiction is obtained as before. Thus $\mathcal{T}$ is isomorphic to the (graph of the) regular octahedron. Note that, by the Gauß–Bonnet formula, no non-obtuse triangle can contain three vertices of $\mathcal{D}$ in its interior.

Suppose at most 3 vertices of $\mathcal{T}$ coincide with vertices of $\mathcal{D}$. Then the remaining at least 17 vertices of $\mathcal{D}$ lie in the interiors of the 8 triangles. So, some triangle must contain at least 3 vertices of $\mathcal{D}$ in its interior, which is impossible. Hence at least 4 vertices v_1, v_2, v_3, v_4 of $\mathcal{T}$ coincide with vertices of $\mathcal{D}$.

Any 4 vertices of the regular octahedron determine 2 triangles or no triangle. If the 4 vertices v_1, v_2, v_3, v_4 of $\mathcal{T}$ determine 2 triangles of $\mathcal{T}$, then, by Remark 1, they contain at most 1 vertex each. So, the remaining 6 triangles of $\mathcal{T}$ must contain at least 14 vertices of $\mathcal{D}$ in their interiors. This implies that some triangle of $\mathcal{T}$ has at least 3 vertices of $\mathcal{D}$ in its interior, which is impossible. Hence the 4 vertices determine a cycle C in $\mathcal{T}$ which decomposes $\mathcal{D}$ into 2 regions D_1 and D_2, each of which contains a further vertex of $\mathcal{T}$. Let $v_{i+4} \in D_i$ be these vertices. By Remark 1, v_5 and v_6 are not vertices of $\mathcal{D}$. Hence 16 vertices of $\mathcal{D}$ are interior to the 8 triangles of $\mathcal{T}$, which means that each triangle has 2 vertices of $\mathcal{D}$ in its interior.

According to Lemma 2, the four triangles in D_i must be of types III, III, III, III, or III, IV, III, IV, in this order ($i = 1, 2$). However, it is clearly impossible for a triangle of type III in D_1 to have a neighbour of type III or IV in D_2. The proof is finished. □

3. Acute triangulations

Theorem 2. *The dodecahedron admits an acute triangulation with 14 triangles and no acute triangulation with less than 12 triangles.*

Proof. Regarding the second assertion of the theorem, it suffices, in view of Theorem 1, to prove that there is no acute triangulation of the dodecahedral surface $\mathcal{D}$ with 10 triangles.

Suppose on the contrary there is such a triangulation $\mathcal{T}$. Then $\mathcal{T}$ must have 15 edges and, by Euler's formula, 7 vertices. The degree of any vertex of $\mathcal{T}$ is at least 4.

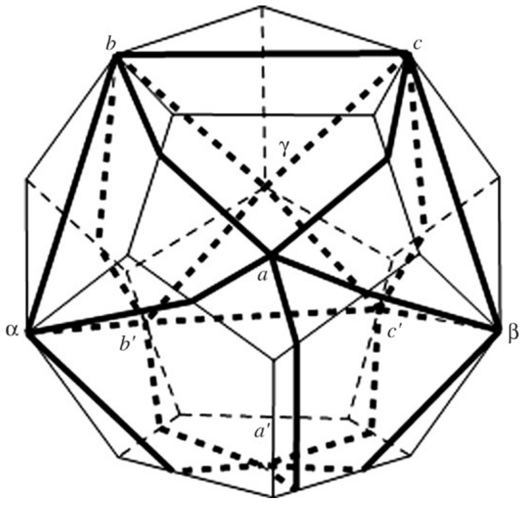

Fig. 9.

It is immediately seen that $\mathcal{T}$ consists of a 5-cycle C each vertex of which has degree 4 plus a point connected to the vertices of C in each of the two domains D_1, D_2 determined by C on $\mathcal{D}$.

Each vertex of $\mathcal{T}$ which is not a vertex of $\mathcal{D}$ must have degree 5. So, five of the vertices of $\mathcal{D}$, say $v_1, \ldots, v_5$ are on C; the remaining 15 are interior to the 10 triangles of $\mathcal{T}$. Hence five triangles must contain precisely two interior vertices each. Obviously, either D_1 or D_2 contains at least three of these triangles. Then two of them, $v_1v_2v_6$ and $v_2v_3v_6$ say, must be neighbours having a common side, v_2v_6, connecting the 4-valent vertex v_2 with the 5-valent vertex v_6. By Lemma 2, they are of types III and IV. They lie as seen on Fig. 7, with v_6 in $b'c'd'e'f'$. Due to the position of v_6 explained in Remark 2 and shown in Fig. 7, the only possible position for $v_5v_1v_6$ is with $v_5 = f$. Now, if $v_4 = b$ then $\angle v_4v_5v_6 + \angle v_6v_5v_1 = \pi$, and if $v_4 = b'$ then $\angle v_3v_4v_6 + \angle v_6v_4v_5 = \pi$, both impossible. □

We present now an acute triangulation $\mathcal{T}$ of $\mathcal{D}$ consisting of 14 triangles. (See Figs. 9 and 10.) We denote the upper pentagon of the dodecahedron by u, the lower pentagon by l, the five pentagons adjacent to u by s_1, s_2, s_3, s_4, s_5, and those adjacent to l by $s'_1, s'_2, s'_3, s'_4, s'_5$, assuming that s_i, s'_i, s_{i+1} have a common vertex and s'_i, s_{i+1}, s'_{i+1} have a common vertex (indices taken modulo 5).

Denote by o_i (resp. o'_i) the centre of s_i (resp. s'_i) for $i = 1, \ldots, 5$, by o_u the centre of u, and by o_l the centre of l.

Let α (resp. β and γ) be the common vertex of the faces s_2, s'_1, s'_2 (resp. s_4, s'_3, s'_4 and s_1, s_5, s'_5). These 3 vertices of $\mathcal{T}$ have degree 4.

Let a (resp. a') be the intersection point of the line segment from $s_3 \cap s'_2 \cap s'_3$ (resp. $s'_2 \cap s'_3 \cap l$) to o_3 (resp. o_l) with a diagonal of s_3 (resp. l). Let $b = s_1 \cap s_2 \cap u$ and $c = s_4 \cap s_5 \cap u$ (we denote the single-point set $\{p\}$ by p, too). Further, let b', c' be the points on $s'_1 \cap s'_5$, respectively $s'_4 \cap s'_5$, such that $b'y_b/b'x_b = c'y_c/c'x_c = 2$), where x_b (resp. x_c) denotes the common vertex of s_1, s'_1, s'_5, (resp. s_5, s'_4, s'_5) and y_b (resp. y_c) the common vertex of the faces s'_1, s'_5, l, (resp. s'_4, s'_5, l).

These 6 vertices of $\mathcal{T}$ have degree 5.

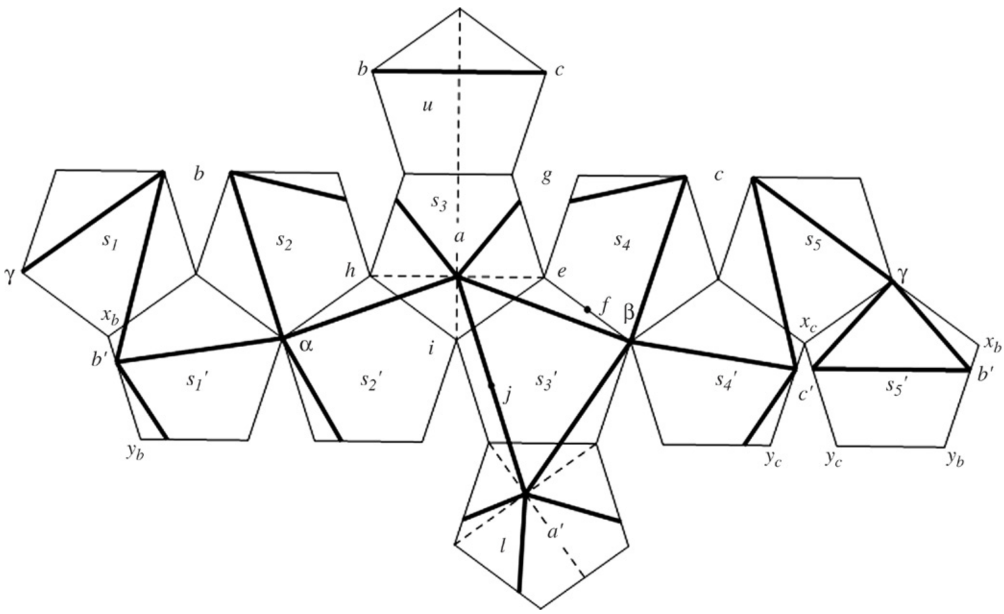

Fig. 10.

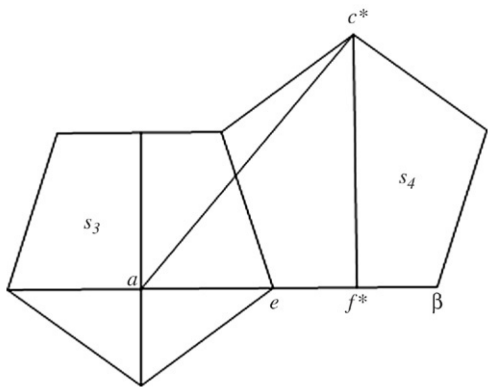

Fig. 11.

We get 14 triangles

$$abc, a'b'c', aa'\alpha, aa'\beta, bb'\alpha, bb'\gamma, cc'\beta, cc'\gamma, ab\alpha, ac\beta, a'b'\alpha, a'c'\alpha, bc\gamma, b'c'\gamma.$$

(Note that there are two shortest paths from a to a'. The edge aa' of $\mathcal{T}$ is one of them.)

Let us show that all triangles are acute.

We start with the triangle abc. First, $\angle cab = 2\angle cao_u$. Put $e = s_3 \cap s_4 \cap s_3'$ and let f be the middle point of the segment $e\beta$. Rotate c and f around $s_3 \cap s_4$ to become coplanar with s_3, the new positions being c^* and f^* respectively. (See Fig. 11.) Note that a, e, f^* are collinear and c^*af^* is a right triangle. Since $co_4 > ae$ and $o_4 f > ef$, we have $cf > ae + ef$, i.e. $c^* f^* > af^*$. Then $\angle cae > \frac{\pi}{4}$, which implies $\angle cao_u < \frac{\pi}{4}$ and $\angle cab < \frac{\pi}{2}$.

Second, $\angle bca = \angle cba = \angle bcg + \angle gca$, where $g = u \cap s_3 \cap s_4$. Since $\angle bcg = \angle \beta cg = \frac{2\pi}{5}$ and $\angle \beta cf = \frac{\pi}{10}$, it suffices to show that $\angle fca > \frac{\pi}{5}$. Indeed,

$$\tan \angle fca > \tan \frac{\pi}{5}$$

because

$$\tan \angle f^* c^* a = \frac{f^* e + ea}{c^* f^*} = \frac{\frac{1}{2} + \cos \frac{\pi}{5}}{\sin \frac{\pi}{5} + \cos \frac{\pi}{10}} > \frac{\sin \frac{\pi}{5}}{\cos \frac{\pi}{5}},$$

which is equivalent to

$$\frac{1}{2} \cos \frac{\pi}{5} + \cos \frac{2\pi}{5} > \sin \frac{\pi}{5} \cos \frac{\pi}{10}$$

and to

$$\frac{1}{2} \cos \frac{\pi}{5} + \sin \frac{\pi}{10} > 2 \sin \frac{\pi}{10} \cos^2 \frac{\pi}{10}.$$

This inequality is true because, putting $\sin \frac{\pi}{10} = \frac{\sqrt{5}-1}{8}$ and $\cos \frac{\pi}{5} = \frac{\sqrt{5}+1}{4}$, we see that $\frac{1}{2} \cos \frac{\pi}{5} > \sin \frac{\pi}{10}$, which yields

$$\frac{1}{2} \cos \frac{\pi}{5} > \sin \frac{\pi}{10} \left(2 \cos^2 \frac{\pi}{10} - 1 \right).$$

Next, let us consider the triangle $a\beta c$. Obviously, $\angle ac\beta < \frac{2\pi}{5}$. Also, $\angle c\beta a = \angle c\beta e + \angle e\beta a = \frac{2\pi}{5} + \angle h\alpha a$, where $h = s_1 \cap s_3 \cap s_2'$. Since the convex increasing function $\tan : [0, \pi/2) \to [0, \infty)$ satisfies $2 \tan \frac{\pi}{10} < \tan \frac{\pi}{5}$ and

$$2 \tan \angle a\alpha i = 2 \frac{ai}{\alpha i} = \frac{ai}{ha} = \tan \frac{\pi}{5},$$

where $i = s_2 \cap s_2' \cap s_3'$, we have $\frac{\pi}{10} < \angle a\alpha i$. Hence $\angle h\alpha a < \frac{\pi}{10}$ and $\angle c\beta a < \frac{\pi}{2}$.

Finally, $\angle ca\beta = \angle cae + \angle ea\beta$ where $\angle cae < \frac{3\pi}{10}$ because $\angle fca > \frac{\pi}{5}$ and $\angle ea\beta < \frac{\pi}{5}$ because a is outside the intrinsic circle circumscribed to s_3' (to which ae is a tangent). Hence $\angle ca\beta < \frac{\pi}{2}$.

Next we consider the triangles $aa'\beta$ and $aa'\alpha$. There are two possibilities for the edge aa': either to go through the face s_2' or through s_3'; let us take it through s_3'. Note that $\angle iaa' = \frac{\pi}{10}$.

We shall show that $\angle a'a\beta < \frac{3\pi}{10}$.

Let j be the mid-point of aa'. Note that $\angle a\beta j = \angle a\beta i + \angle i\beta j$, where $\angle a\beta i = \angle a\alpha i > \frac{\pi}{10}$ and $\angle i\beta j = \frac{\pi}{10}$. Then $\angle a\beta j > \frac{\pi}{5}$ and $\angle a'a\beta < \frac{3\pi}{10}$.

Also,

$$\angle \alpha aa' = \angle \alpha a'a = \angle a'a\beta + 2 \cdot \frac{\pi}{10} < \frac{\pi}{2}.$$

In the triangle $a\beta j$, $aj = \frac{1}{2} + \sin \frac{\pi}{5} \cos \frac{\pi}{10}$ and $\beta j = \sin \frac{\pi}{5} + \cos \frac{\pi}{10} - \sin \frac{\pi}{10}$. We have $aj < \beta j$ because $\sin \frac{\pi}{5} \cos \frac{\pi}{10} < \sin \frac{\pi}{5}$ and $\frac{1}{2} < \cos \frac{\pi}{10} - \sin \frac{\pi}{10}$. To see the second inequality, we get by squaring

$$\frac{1}{4} < 1 - 2 \sin \frac{\pi}{10} \cos \frac{\pi}{10} = 1 - \sin \frac{\pi}{5},$$

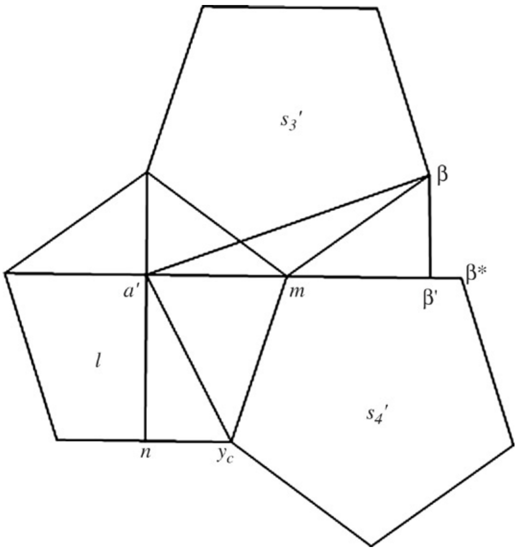

Fig. 12.

which follows from

$$\sin\frac{\pi}{5} < \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} < \frac{3}{4}.$$

Then $\angle a\beta j < \frac{\pi}{4}$ and $\angle a\beta a' < \frac{\pi}{2}$.

Next let us consider the triangle $c\beta c'$. Clearly, $\angle \beta cc' < \frac{2\pi}{5}$. Note that $\angle c\beta c' = \angle c\beta x_c + \angle x_c\beta c'$, where $\angle c\beta x_c = \frac{2\pi}{5}$ and $\angle x_c\beta c' < \frac{\pi}{10}$. Also, $c\beta < cc'$ implies $\angle cc'\beta < \angle c\beta c' < \frac{\pi}{2}$.

We now look at the triangle $a'\beta c'$. It is trivial that $\angle a'\beta c' < \angle a\beta c < \frac{\pi}{2}$ and $\angle \beta c'a' < \angle \beta c' y_c < \pi/2$.

Clearly, $\angle \beta a'c' < \angle \beta a' y_c$. Rotate β around $l \cap s_4'$ to become coplanar with l, the new position being β^*. Also, rotate β around $l \cap s_3'$ to become coplanar with l and denote it again by β. Consider the projection β' of β on $\beta^* a'$. (See Fig. 12.) Regarding the triangles $\beta\beta' a'$ and $y_c n a'$, where n is the middle point of the segment $l \cap s_5'$, if the inequality $\beta\beta'/\beta' a' < y_c n/na'$ holds then $\angle \beta a' y_c < \frac{\pi}{2}$; and indeed,

$$\frac{\sin\frac{\pi}{5}}{2\cos\frac{\pi}{5}} < \frac{\frac{1}{2}}{\cos\frac{\pi}{10}},$$

because

$$2\sin\frac{\pi}{10}\cos^2\frac{\pi}{10} < 2\cos^2\frac{\pi}{10} - 1$$

i.e.,

$$2\cdot\frac{\sqrt{5}-1}{4}\cdot\frac{5+\sqrt{5}}{8} < 2\cdot\frac{5+\sqrt{5}}{8} - 1,$$

which becomes

$$\frac{\sqrt{5}}{4} < \frac{1+\sqrt{5}}{4}.$$

Next let us consider the triangle $cc'\gamma$. Clearly, $\angle c'c\gamma < \frac{2\pi}{5}$. Also, $\angle c\gamma c' = \frac{2\pi}{5} + \angle x_c\gamma c'$. From the inequality

$$\frac{y_c\gamma}{x_c\gamma} < 2 = \frac{y_c c'}{x_c c'}$$

it follows that $\angle x_c\gamma c' < \angle c'\gamma y_c$, whence $\angle x_c\gamma c' < \frac{\pi}{10}$ and $\angle c\gamma c' < \frac{\pi}{2}$.

Let δ be the angle between cc' and o_5x_c. We have

$$\tan\delta = \frac{\frac{1}{2} - \frac{1}{3}\sin\frac{\pi}{10}}{\sin\frac{\pi}{5} + \cos\frac{\pi}{10} + \frac{1}{3}\cos\frac{\pi}{10}} < \frac{\frac{1}{3}\cos\frac{\pi}{10}}{\frac{1}{3}\sin\frac{\pi}{10} + 1} = \tan\angle x_c\gamma c',$$

because

$$\frac{1}{2} - \frac{1}{6}\sin\frac{\pi}{10} < \frac{1}{3}\sin\frac{\pi}{5}\cos\frac{\pi}{10} + \frac{1}{9} + \frac{1}{3}\cos^2\frac{\pi}{10},$$

i.e.,

$$\frac{1}{2} - \frac{1}{6}\cdot\frac{\sqrt{5}-1}{8} < \frac{1}{3}\cdot\frac{\sqrt{5}}{8} + \frac{1}{9} + \frac{1}{3}\cdot\frac{5+\sqrt{5}}{8},$$

which reduces to $32 < 15\sqrt{5}$.

Hence $\delta < \angle x_c\gamma c'$ and therefore $\angle cc'\gamma < \frac{\pi}{2}$.

The triangle γbc is equilateral, with all angles equal to $\frac{2\pi}{5}$.

Next let us consider the triangle $\gamma b'c'$. Obviously, $\angle\gamma c'b' = \angle\gamma b'c' < \frac{2\pi}{5}$.

We now prove that $\angle x_c\gamma c' > \frac{\pi}{20}$. First observe that $\angle x_c\gamma s > \frac{\pi}{10}$, where s is the midpoint of x_cy_c. Take a point c^+ on the edge x_cy_c such that $\angle x_c\gamma c^+ = \angle c^+\gamma s$. Since $x_c\gamma < \gamma s$, we have $x_cc^+ < c^+s$. On the other side, by the choice of c', $x_cc' > c's$. It follows that $\angle x_c\gamma c' > \frac{\pi}{20}$. Hence $\angle b'\gamma c' < \frac{\pi}{2}$.

Finally, let us consider the triangle $a'b'c'$.

We will show that $\angle b'c'a' < \frac{\pi}{2}$ (see Fig. 13). Let z be the vertex $s_1' \cap s_2' \cap l$ and w the vertex $s_2' \cap s_3' \cap l$. Choose v' on the arc wz of the circle C circumscribed to l such that $\angle v'o_lw = \frac{\pi}{10}$ and choose p' on o_lv' such that $p'o_l/p'v' = 2$. Let q' be the orthogonal projection of p' on o_lw. In the quadrangle $x_cy_co_lv'$ the angles at y_c and o_l are equal and $x_cy_c > o_lv'$. Hence $\angle o_lp'c' > \frac{\pi}{10}$. Suppose that $o_la' < a'p'$ (this will be shown later). It will follow that $\angle o_lp'a' < \frac{\pi}{10} < \angle o_lp'c'$, which implies $\angle y_cc'a' < \angle y_cc'p' < \frac{\pi}{10}$. This yields $\angle b'c'a' < \frac{\pi}{2}$.

Now, we prove $o_la' < a'p'$. We have $o_la' = R\sin\frac{\pi}{10}$, where R is the radius of C. Also,

$$\frac{p'o_l}{v'o_l} = \frac{2}{3}$$

and

$$a'p'^2 = (o_lq' - o_la')^2 + q'p'^2 = \frac{4}{9}R^2 - \frac{4}{3}R^2\sin\frac{\pi}{10}\cos\frac{\pi}{10} + R^2\sin^2\frac{\pi}{10}.$$

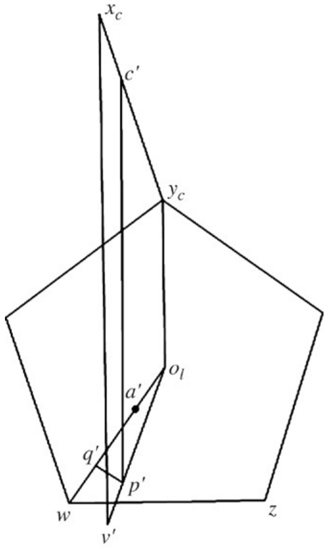

Fig. 13.

And indeed,

$$\sin^2 \frac{\pi}{10} < \frac{4}{9} - \frac{2}{3} \sin \frac{\pi}{5} + \sin^2 \frac{\pi}{10}$$

since

$$\sin \frac{\pi}{5} < \frac{2}{3}.$$

Concerning the angle at a', obviously $\angle b'a'c' < \angle bac < \frac{\pi}{2}$. □

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References

[1] B.S. Baker, E. Grosse, C.S. Rafferty, Nonobtuse triangulation of polygons, Discrete Comput. Geom. 3 (1988) 147–168.

[2] Y.D. Burago, V.A. Zalgaller, Polyhedral embedding of a net, Vestn. Leningrad. Univ. 15 (1960) 66–80 (in Russian).

[3] Ch. Cassidy, G. Lord, A square acutely triangulated, J. Recr. Math. 13 (1980) 263–268.

[4] P.G. Ciarlet, P.A. Raviart, Maximum principle and uniform convergence for the finite element method, Comput. Methods Appl. Mech. Engrg. 2 (1973) 17–31.

[5] Y. Colin de Verdière, Comment rendre géodésique une triangulation d'une surface? Enseign. Math., II. Sér. 37 (1991) 201–212.
[6] Y. Colin de Verdière, A. Marin, Triangulations presque équilaterales des surfaces, J. Differential Geom. 32 (1990) 199–207.
[7] I. Fary, On straight line representation of planar graphs, Acta Sci. Math. (Szeged) 11 (1948) 229–233.
[8] M. Gardner, Mathematical games. A fifth collection of brain-teasers, Sci. Am. 202 (2) (1960) 150–154.
[9] M. Gardner, Mathematical games. The games and puzzles of Lewis Carroll, and the answers to February's problems, Sci. Am. 202 (3) (1960) 172–182.
[10] J.L. Gerver, The dissection of a polygon into nearly equilateral triangles, Geom. Dedicata 16 (1984) 93–106.
[11] T. Hangan, J.-i. Itoh, T. Zamfirescu, Acute triangulations, Bull. Math. Soc. Sci. Math. Roumanie 43 (2000) 279–286.
[12] J.-i. Itoh, Acute triangulations of sphere and icosahedron, Josai Math. Monographs 3 (2001) 53–61.
[13] J.-i. Itoh, T. Zamfirescu, Acute triangulations of the regular icosahedral surface, Discrete Comput. Geom. 31 (2004) 197–206.
[14] M. Křížek, L. Qun, On diagonal dominance of stiffness matrices in 3D, East–West J. Numer. Math. 3 (1) (1995) 59–69.
[15] S. Korotov, M. Křížek, Acute type refinements of tetrahedral partitions of polyhedral domains, manuscript, 2001.
[16] S. Korotov, M. Křížek, P. Neittaanmäki, Weakened acute type condition for tetrahedral triangulations and the discrete maximum principle, University of Jyväskylä, Report B 11/1999.
[17] R.H. MacNeal, An asymmetrical finite difference network, Quart. Appl. Math. 11 (1953) 295–310.
[18] H. Maehara, On acute triangulations of quadrilaterals, in: Akiyama, Jin et al. (Eds.), Discrete and Computational Geometry. Japanese Conference, JCDCG 2000, Tokyo, Japan, 22–25 November, 2000. Revised Papers. Berlin: Springer. Lect. Notes Comput. Sci. 2098 (2001) 237–243.
[19] H. Maehara, Acute triangulations of polygons, European J. Combin. 23 (2002) 45–55.
[20] V.R. Santos, On the strong maximum principle for some piecewise linear finite element approximate problems of non-positive type, J. Fac. Sci. Univ. Tokyo, Sect. IA Math. 29 (1982) 473–491.
[21] G. Strang, G.J. Fix, An Analysis of the Finite Element Method, Prentice-Hall, Englewood Cliffs, NJ, 1973.
[22] W. Tutte, How to draw a graph, Proc. London Math. Soc. 13 (1963) 743–768.
[23] L. Yuan, Acute triangulations of polygons, Discrete Comput. Geom. 34 (2005) 697–706.
[24] C. Zamfirescu, Acute triangulations of the double triangle, Bull. Math. Soc. Sci. Math. Roumanie 47 (2004) 189–193.
[25] T. Zamfirescu, Acute triangulations: a short survey, in: Proc. 6th Annual Conference Romanian Soc. Math. Sciences I, 2002, pp. 10–18.