Adv. Geom. 7 (2007), 385–390 **Advances in Geometry**
DOI 10.1515/ADVGEOM.2007.24

Antipodal trees and mutually critical points on surfaces

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(Communicated by K. Strambach)

Abstract. To any point on an Alexandrov surface homeomorphic to the sphere one can associate a minimal subtree of the cut locus containing all farthest points. It is called the antipodal tree.

Two points of a compact orientable Alexandrov surface are called mutually critical if each of them is critical with respect to the other. All points which are mutually critical with a given point form a set. In this paper we show that this set, as well as the set of endpoints of any antipodal tree, are finite.

2000 Mathematics Subject Classification. 53C45, 53C22

Introduction

Let us recall the definition of an Alexandrov space (with curvature bounded below). If a, b, c belong to the metric space $(\mathcal{A}, \rho)$, let $\angle^k abc$ denote the angle of the geodesic triangle in S_k of side-lengths $\rho(a, b)$, $\rho(b, c)$, $\rho(c, a)$, opposite to the side of length $\rho(c, a)$, where S_k denotes the 2-dimensional complete simply-connected Riemannian manifold of constant curvature $k < 0$ (a Lobachevskii plane).

Here, an *Alexandrov space* is a complete metric space $(\mathcal{A}, \rho)$ such that any pair of points in $\mathcal{A}$ admits a midpoint (so, the metric is intrinsic), and every point of $\mathcal{A}$ has a neighbourhood in which, for any four distinct points a, b, c, d, we have

$$\angle^k bac + \angle^k cad + \angle^k dab \leq 2\pi.$$

The original definition in [2] is slightly more general, not assuming completeness of $(\mathcal{A}, \rho)$.

Burago, Gromov and Perelman [2] also introduced and investigated the notion of dimension in Alexandrov spaces. Any 2-dimensional Alexandrov space is a topological 2-manifold. If nothing else is specified, this is the meaning of a *surface*.

*Work started in 2004 during a research stay at MSRI, Berkeley, California, the generous support of which is thankfully acknowledged.

Let S be a surface (with intrinsic metric ρ). For an arbitrary point $x \in S$ we consider the distance function $\rho_x(y) = \rho(x, y)$ from x to $y \in S$ and the *cut locus* $C(x)$, defined as the set of all $y \in S \setminus \{x\}$ such that no *segment*, i.e., shortest path, from x to y can be extended as a segment beyond y.

The notion of a cut locus was first considered by H. Poincaré [7] in 1905. Among other things, it is known that (on a surface) the cut locus is locally a tree. For an introduction to the cut locus see for example [5]. A detailed description of the cut locus on surfaces was given by Shiohama and Tanaka [8].

A point of a tree is called *endpoint* if its deletion does not disconnect the tree. A tree is *finite* if it has finitely many endpoints.

Several interesting sets associated to x lie in the cut locus $C(x)$. For example, the set $Q(x)$ of all critical points with respect to x and ρ_x. Among these, of special importance for the global investigation of a surface S is the set F_x of all absolute maxima of ρ_x. However, other interesting sets of critical points, like the set M_x of all relative maxima of ρ_x, have also been considered.

Let σ, τ be segments on S with a common endpoint x, and take $s \in \sigma$, $t \in \tau$. Then the angle of σ, τ at x is denoted by $\angle\sigma x\tau$, or simply by $\angle\sigma\tau$ if σ and τ have no other common endpoint. The same angle is also denoted by $\angle sxt$ if it is clear from the context that the involved segments from x to s and t are along σ and τ, respectively.

A point $y \in S$ is called *critical* with respect to x (and ρ_x) if for any direction δ at y there exists a segment from y to x making with δ an angle at y not larger than $\pi/2$ (see, for instance, [3], p. 2).

If the surface S is homeomorphic to S^2, then every cut locus is a tree. For $x \in S$, the minimal subtree of $C(x)$ containing F_x is then called the *antipodal tree* of x. Notice that this term has been defined differently in [14]; now we prefer this meaning. We may encounter uncountable sets F_x on one hand (see [12], p. 320) and, on the other, $C(x)$ may be quite large, even residual in S [11].

We pointed out in [15] that in fact $Q(x)$ cannot be too scattered in $C(x)$ if S is a Riemannian surface; more precisely, if S is orientable, it must belong to a single finite tree in $C(x)$ the number of endpoints of which depends on the positive Gauß curvature and on the genus of S.

The case of a convex surface was treated in [14] without any differentiability assumptions. It was found that $Q(x)$ always lies in a subtree of $C(x)$ with at most 4 endpoints. About F_x, still in the convex case, we showed in [13] that every antipodal tree is a Jordan arc or a point.

We will show here that, on any surface homeomorphic to S^2, the antipodal trees are finite. This is a contribution to the description of F_x for which H. Steinhaus has asked (see [4]).

For any compact surface S, let diam$S$ denote the largest value of $\rho(x, y)$ as x and y run through S. Two points realizing diam$S$ are called *diametrically opposite* to each other. There might be no point diametrically opposite to some point $x \in S$, but there might also be several points diametrically opposite to x. The set of all of them is called the *diametrically opposite set* of x. When it is not empty, this set equals F_x.

In fact, the set F_x itself can be a Jordan arc: consider a doubly covered half-disc. We shall show that this is not possible on smooth orientable surfaces, if F_x is a diametrically opposite set. More precisely, F_x must then be finite. Moreover, we shall extend this result to any set of points y, each of which is *mutually critical with* x, i.e., y is critical with respect to x, and vice-versa.

A *geodesic triangle* is a triangle with segments as sides.

If $x \in S$, then the space Σ_x of directions at x is known to have length at most 2π (see [2], p. 23). If Σ_x has length 2π for all $x \in S$, we call S *smooth.* Note that, for example when S is convex, smoothness excludes no singular points except the conical ones, so it admits non-C^1 surfaces.

Antipodal trees

The following lemma is the 2-dimensional case of a generalized form of Toponogov's comparison theorem (see [2], p. 7). We associate here Pizzetti's name to it, too, because for (smooth) surfaces it was Pizzetti who gave it first, in a series of papers at the beginning of the last century (see [6]).

Lemma 1 (Pizzetti–Toponogov). *For any geodesic triangle abc in S,*

$$\angle^k abc \le \angle abc, \quad \angle^k bca \le \angle bca, \quad \angle^k cab \le \angle cab.$$

The following lemma is well-known in various contexts. For surfaces of class C^2, see [9].

Lemma 2 (Lemma 9 in [10]). *For any $x \in S$, we have $M_x \subset Q_x$.*

We use these lemmas to prove our first result.

Theorem 1. *Any antipodal tree on a surface homeomorphic to* S^2 *is finite.*

Proof. Suppose the antipodal tree T_x of x has an infinite set E of endpoints. Select a convergent subsequence $\{y_n\}_{n=1}^{\infty}$ with $y_n \in E$ and $y_n \to y$. Clearly, $E \subset F_x$ and $y \in T_x$. Let Y_n be the union of all segments from x to y_n, and D_n the component of $S \setminus Y_n$ containing $T_x \setminus \{y_n\}$. Let σ_n, σ'_n be the (possibly coinciding) segments from x to y_n lying in bd D_n, and α_n the angle at y_n between σ_n and σ'_n towards D_n.

Take the points $x', y'_n, y'_m \in S_k$ satisfying $\rho'(x', y'_n) = \rho(x, y_n)$, $\rho'(x', y'_m) = \rho(x, y_m)$ and $\rho'(y'_n, y'_m) = \rho(y_n, y_m)$, where ρ' is the (standard) metric of S_k. Then the triangle $x'y'_n y'_m$ is isosceles and $\angle x'y'_n y'_m \to \pi/2$ as $n, m \to \infty$.

Let τ_{nm} be a segment from y_n to y_m. By Pizzetti–Toponogov's comparison theorem (our Lemma 1), for arbitrary $\varepsilon > 0$, $\angle \sigma_n \tau_{nm} > (\pi/2) - \varepsilon$ and $\angle \sigma'_n \tau_{nm} > (\pi/2) - \varepsilon$ if n, m are large enough. Similarly, if τ_n is a segment from y_n to y, then $\angle \sigma_n \tau_n > (\pi/2) - \varepsilon$ and $\angle \sigma'_n \tau_n > (\pi/2) - \varepsilon$ for n large enough.

By Lemma 2, y_n is a critical point with respect to x, so $\alpha_n \le \pi$. Now,

$$\alpha_n = \angle \sigma_n \tau_n + \angle \sigma'_n \tau_n = \angle \sigma_n \tau_{nm} + \angle \sigma'_n \tau_{nm}.$$

Also, $\angle\sigma_n\tau_n < (\pi/2) + \varepsilon$, otherwise $\angle\sigma'_n\tau_n = \alpha_n - \angle\sigma_n\tau_n \leq \pi - \varepsilon$ and a contradiction is obtained. Similarly, $\angle\sigma_n\tau_{nm} < (\pi/2) + \varepsilon$.

It follows that $\angle\tau_n\tau_{nm} < 2\varepsilon$. Thus, in the triangle $y_n y_m y_k$, for large indices n, m, k, we have

$$\angle y_m y_n y_k \leq \angle\tau_n\tau_{nm} + \angle\tau_n\tau_{nk} < 4\varepsilon.$$

Analogously, both other angles of the triangle $y_n y_m y_k$ are less than 4ε. But the sum of the angles of the triangle on S_k with the same side-lengths as $y_n y_m y_k$ tends to π as $n, m, k \to \infty$. By Lemma 1, the sum of the angles of $y_n y_m y_k$ exceeds $\pi - \varepsilon$ for indices large enough, which contradicts for small ε the previous findings.

Mutually critical points

We start with another lemma from [2].

Lemma 3 ([2], p. 6). *Let $a_n \to a$, $b_n \to b$, $c_n \to c$ on the surface S. Then*

$$\liminf \angle a_n b_n c_n \geq \angle abc.$$

We have seen in the preceding section that the antipodal tree T_x of $x \in S$ must be finite if S is homeomorphic to a sphere. But F_x does not need to be a finite set, even in the convex case. It must, however, be finite if it is a diametrically opposite set and S is smooth, as we shall see.

Theorem 2. *Let S be a smooth orientable surface and $x \in S$. Then the set of all points in S mutually critical with x is finite.*

Proof. Suppose the point x has an infinite set Y of points mutually critical with x. Then we may choose a convergent sequence of such points, $y_n \to y$, where $y \in Y$ too, because Y is closed. (This follows from Lemma 3.)

Since $x \in Q(y_n)$, there are three segments from x to y_n making pairwise at x angles which are less than π but sum up to 2π, or there are two segments from x to y_n making at x the angle π. Since $y_n \in Q(x)$, there also exist three segments from x to y_n making pairwise at y_n angles less than π summing up to 2π, or there exist two segments from x to y_n making at x the angle π.

Let $\mathcal{F}$ be the set of all segments which are limits of segments from x to y_n.

First suppose that there are three distinct limit segments $\sigma^1, \sigma^2, \sigma^3$ from x to y in $\mathcal{F}$. Let σ_n^i be segments from x to y_n such that $\sigma_n^i \to \sigma^i$ $(i = 1, 2, 3)$. There are three sectors of S, locally at y, determined by $\sigma^1, \sigma^2, \sigma^3$. Again for a subsequence of indices, y_n lies in one of these sectors, say determined by σ^1, σ^2. But $\sigma_n^3 \to \sigma_3$, whence σ_n^3 necessarily crosses $\sigma_1 \cup \sigma_2$ for large n, which is impossible.

Now suppose that $\mathcal{F}$ contains precisely two limit segments σ^1, σ^2. In this case, the above angle conditions imply that σ^1, σ^2 make the angle π at x. Let σ_n be an arbitrary segment from x to y_n. Take a tubular neighbourhood N of $\sigma^1 \cup \sigma^2$. For $\{\sigma_n\}_{n=1}^{\infty}$, the only possible limit segments are σ^1 and σ^2. So, for n large enough, $\sigma_n \subset N$. Since S

is orientable, $N \setminus (\sigma^1 \cup \sigma^2)$ is disconnected, whence, for each n, $\sigma_n \setminus \{x\}$ lies on only one side of $\sigma^1 \cup \sigma^2$ locally at x. Hence this happens, for infinitely many indices, with the same side. Let δ be the direction at x orthogonal to $\sigma^1 \cup \sigma^2$ and pointing to the other side of $\sigma^1 \cup \sigma^2$. Then the angle between δ and σ_n is larger than $\pi/2$, for infinitely many indices and any choice of σ_n. This contradicts $x \in Q(y_n)$.

Corollary. *Let S be a smooth orientable surface. Then the diametrically opposite set of any point in S is finite.*

This is an immediate consequence of Theorem 2. In particular, if S has genus 0, each non-empty diametrically opposite set is single-valued [10], and, if S has genus 1, any diametrically opposite set has at most 5 points [1].

The corollary cannot be extended to non-orientable surfaces, as the example of the standard projective plane shows. Smoothness is equally necessary: consider the convex hull of $\{(0,0)\} \cup \{(\cos\varphi, \sin\varphi) : 0 \le \varphi \le 1\}$, doubly covered.

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Received 21 October, 2005; revised 24 February, 2006

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