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Viewing and realizing diameters

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Abstract. This paper is about diameters of compact sets. These are chords of maximal length. On one hand we see that the sum of the angles under which we see a diameter from two points of the set separated by the diameter is never smaller than $5\pi/6$. On the other hand we describe cases in which the diameter of a point-symmetric set must join symmetric points.

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1. Introduction

Let K be a compact set in the Euclidean plane $\mathbb{R}^2$. There is at least one pair of points $x, y \in K$ such that

$$\|x - y\| = \sup_{u,v\in K} \|u - v\|.$$

We call the line-segment xy a *diameter* of K. It is immediately seen that no point of $K \setminus xy$ sees xy under an angle less than $\pi/3$.

We shall say that a pair of points sees a line-segment under the angle α if the sum of the two angles under which they see the line-segment equals α. In this paper we shall show that the pairs of points separated by the line containing a diameter sees that diameter under an angle larger than $5\pi/6$. Then, we extend this to the case of convex surfaces.

By the Knuth-Kotani puzzle, on the surface of a 1-by-1-by-2 box in $\mathbb{R}^3$, if x is a vertex then the point farthest from x is not the antipodal vertex. Starting from this, J. Propp still conjectured that, on point-symmetric compact convex surfaces, "the maximum distance between two points in the surface metric, as both of them vary over the surface, is achieved by a pair of antipodal points". This has been proven by C. Vîlcu in 2000. We offer here further extensions.

2. Definitions and notation

Throughout this paper (X, ρ) will be a metric space with *intrinsic metric*. This means that every pair of points $x, y \in X$ is joined by an *arc* (i.e., a continuous image of [0, 1]) of length $\rho(x, y)$, called *segment*. Note that the segment joining two points may not be unique. Let

K be a compact subspace of X equipped with the metric induced by ρ (denoted by ρ too). A segment in X with endpoints in K and of maximal length is called a *diameter* of K. Its length is denoted by diam$K$. A triangle with segments as sides is called *geodesic*.

Let σ be a diameter of K. A pair of points $u, v \in K \setminus \sigma$ is said to be a *σ-separated pair* if some segment uv from u to v meets σ. We say (in spaces where angles are defined) that the pair (u, v) *sees σ under the angle α* if the sum of the two angles under which u and v see σ equals α.

We also say that ρ is *nonbifurcating* if for any pair of segments starting at the same point and having a second common point x, either x is the other common endpoint, or one of the segments includes the other.

3. Viewing compact sets in Hilbert spaces

In the introduction we considered diameters of compact sets in the plane, which was justified by the essentially 2-dimensional character of the problem.

Let X be a Hilbert space of dimension at least 2 and ρ be its usual norm-distance. Thus, the segments in X are the usual line-segments.

THEOREM 1. *For any diameter xy of the compact set $K \subset X$, every xy-separated pair in K sees xy under an angle not less than $5\pi/6$.*

Proof. Let diam$K = 1$. Consider in K an xy-separated pair (u, v). From now on in this proof, we work in the plane Π determined by x, y, u, v. The set $K \cap \Pi$ lies in the intersection D of the closed unit discs of centres x and y. Let C be the closed disc of diameter xy. Let a, b be the angular points of bd$D$ such that u and a lie on the same side of the line $L \supset xy$, and let a_x, a_y be the midpoints of the arcs joining a with x, y in bd$D$. The circle through x, y, a_y, a_x cuts one of the components of $D \setminus C$ into two pieces the closure of which will be denoted by A and B (with $a \in A$). Analogously, we define the sets A', B' symmetric to A, B with respect to L.

If $v \in C$, $\angle xuy + \angle xvy \geq \frac{5}{6}\pi$, because $\angle xuy \geq \frac{1}{3}\pi$ and $\angle xvy \geq \frac{1}{2}\pi$.

If $u \in B$ and $v \in B'$, $\angle xuy \geq \frac{5}{12}\pi$ and $\angle xvy \geq \frac{5}{12}\pi$, which again imply $\angle xuy + \angle xvy \geq \frac{5}{6}\pi$.

If $u \in A$ and $v \in A'$, then u is a_x or a_y and v is the point a'_x or a'_y, symmetric to a_x or a_y with respect to L, otherwise $\|u - v\| > 1$. Hence $\angle xuy + \angle xvy = \frac{5}{6}\pi$.

The only case which remains to be considered is essentially $u \in A$, $v \in B' \setminus C$.

Assume, to make a choice, that v is not separated from y by the line L' through a, b. The circle Γ through v, x, y meets B at x and y only. The circle through v of centre u meets the subarc of bd$D$ from y to a'_y at a point v' outside of Γ or, if $v \in$ bd$D$, on Γ. Thus, $\angle xv'y \leq \angle xvy$.

The circle through u of centre v' meets the subarc of bd$D$ from a to a_y at a point u' farther from any point of $L' \cap C$ than u and therefore outside the circle through x, y, u, if $u \notin$ bd$D$; otherwise $u = u'$. Hence $\angle xu'y \leq \angle xuy$.

Obviously,

$$\|u' - v'\| = \|u - v'\| = \|u - v\| \leq 1,$$

whence $\angle u'xv' \leq \frac{1}{3}\pi$. Putting $\alpha = \angle v'xy$ and $\beta = \angle u'xv'$, we calculate

$$\angle xu'v' = \angle xv'u' = \frac{\pi - \beta}{2},$$

$$\angle xu'y = \angle xu'v' + \angle yu'v' = \frac{\pi - \beta}{2} + \frac{\alpha}{2},$$

$$\angle xv'y = \angle xv'u' + \angle yv'u' = \frac{\pi - \beta}{2} + \frac{\beta - \alpha}{2}.$$

It follows that

$$\angle xuy + \angle xvy \geq \angle xu'y + \angle xv'y = \pi - \frac{\beta}{2} \geq \frac{5}{6}\pi.$$

□

4. Viewing compact sets on convex surfaces

Now, let X be a convex surface in $\mathbb{R}^3$, equipped with the intrinsic distance ρ derived from the metric induced by the Euclidean distance of $\mathbb{R}^3$. This means that $\rho(x, y)$ equals the infimum of all lengths of arcs joining x to y, the lengths being defined using the usual Euclidean distance in $\mathbb{R}^3$. The degenerate case of a doubly covered 2-dimensional convex body is not excluded.

Again, every diameter σ of a compact set $K \subset X$ is seen from any point of $K \setminus \sigma$ under an angle not less than $\frac{1}{3}\pi$. Astonishingly, the result from the previous section can be reproduced without changes.

We first need two lemmas.

LEMMA 1. *(*Pizzetti-Alexandrov's comparison theorem.*) The angles of any geodesic triangle in X are not smaller than the corresponding angles of the Euclidean triangle with the same side-lengths.*

For a proof see [1], p. 132. We associated here Pizzetti's name to it, too, because for smooth surfaces it was Pizzetti who gave it first, in a series of papers at the beginning of the last century (see [3]).

LEMMA 2. *Let* $abc \subset X$ *and* $a_0b_0c_0 \subset \mathbb{R}^2$ *be two triangles as in Lemma* 1. *If* $d \in bc$, $d_0 \in b_0c_0$ *and* $\rho(b, d) = \|b_0 - d_0\|$, *then* $\rho(a, d) \geq \|a_0 - d_0\|$.

This is an immediate consequence of Alexandrov's "Konvexitätsbedingung" ([1], p. 130).

THEOREM 2. *For any diameter* xy *of the compact subset* K *of the convex surface* X, *every* xy*-separated pair in* K *sees* xy *under an angle not less than* $5\pi/6$.

Proof. Let (u, v) be an xy-separated pair in K, and $\{w\} = uv \cap xy$. Consider the points x', y', u', v', w' in $\mathbb{R}^2$ such that $w' \in x'y'$, $x'y' \cap u'v' \neq \emptyset$ (it suffices to choose u', v' separated by the line through x' and y', because they must lie in the intersection of the discs of radii diam$K$ centred at x' and y'), $\|x' - y'\| = \rho(x, y)$, $\|u' - x'\| = \rho(u, x)$, $\|u' - y'\| = \rho(u, y)$, $\|v' - x'\| = \rho(v, x)$, $\|v' - y'\| = \rho(v, y)$, $\|x' - w'\| = \rho(x, w)$. By Lemma 2, $\|u' - w'\| \leq \rho(u, w)$ and $\|v' - w'\| \leq \rho(v, w)$.

By Lemma 1,

$$\angle xuy \geq \angle x'u'y', \qquad \angle xvy \geq \angle x'v'y'.$$

But

$$\|u' - v'\| \leq \|u' - w'\| + \|w' - v'\| \leq \rho(u, w) + \rho(v, w) = \rho(u, v) \leq \rho(x, y) = \|x' - y'\|.$$

By Theorem 1, $\angle x'u'y' + \angle x'v'y' \geq \frac{5}{6}\pi$, whence $\angle xuy + \angle xvy \geq \frac{5}{6}\pi$. □

The following are straightforward consequences of Theorem 2.

COROLLARY 1. *For any diameter* xy *of the compact convex surface* X, *every* xy*-separated pair* (u, v) *sees* xy *under an angle not less than* $5\pi/6$.

COROLLARY 2. *Let* $\mathcal{F}$ *be a family of diameters of the compact convex surface* X *joining the points* $x, y \in X$. *If the points* u, v *lie in different components of* $X \setminus \cup\mathcal{F}$, *then the pair* (u, v) *sees any diameter in* $\mathcal{F}$ *under an angle not less than* $5\pi/6$.

5. Antipodal realizations

In this section let H be a normed linear space and X a compact subset of H, symmetric about the origin $\mathbf{0}$ of H, and admitting an intrinsic metric ρ derived from the metric of X induced by the norm of H. So, any two points $x, y \in X$ are joined by a segment in X. From now on, (X, ρ) will always be a metric space of this kind.

The points $x, -x \in H$ are said to be *opposite*. A diameter is called *antipodal* if its endpoints are opposite.

The following lemma is similar in spirit to, but different from, Lemma 1 in [5].

LEMMA 3. *Let a, b, c, d be pairwise distinct points of a metric space (Y, δ) with δ intrinsic and nonbifurcating. If*

$$\min\{\delta(a, b), \delta(c, d)\} \geq \max\{\delta(a, c), \delta(b, d)\}$$

then no segment from a to c intersects any segment from b to d.

Proof. Assume on the contrary that $ac \cap bd = \{q\}$. Then $\delta(a, q) + \delta(b, q) \geq \delta(a, b)$ and $\delta(c, q) + \delta(d, q) \geq \delta(c, d)$ yield $\delta(a, c) + \delta(b, d) \geq \delta(a, b) + \delta(c, d)$.

This together with the hypothesis implies

$$\delta(a, q) + \delta(b, q) = \delta(a, b) = \delta(a, c),$$

contrary to the assumption that δ is nonbifurcating. □

THEOREM 3. *Let X be homeomorphic to the 2-dimensional standard sphere. If ρ is nonbifurcating then each diameter of X is antipodal.*

Proof. It is rather obvious that $\mathbf{0} \notin X$. Let $x, y \in X$ verify $y \neq -x$ and $\rho(x, y) = \operatorname{diam} X$, and choose a segment σ_{xy} from x to y, and a segment σ_x from x to $-x$. By the symmetry of X, $\rho(-x, -y) = \operatorname{diam} X$, $-\sigma_x$ is a segment from x to $-x$, and $-\sigma_{xy}$ is a segment from $-x$ to $-y$. If $\sigma_x = -\sigma_x$ then $\mathbf{0} \in \sigma_x$ (see the proof of Theorem 4), which contradicts $\mathbf{0} \notin X$. So $\sigma_x \neq -\sigma_x$.

Clearly, $y \notin \sigma_x \cup (-\sigma_x)$. Because in X geodesics do not bifurcate, $\sigma_x \cap (-\sigma_x) = \{x, -x\}$. Since X is homeomorphic to S^2, $\sigma_x \cup (-\sigma_x)$ decomposes X into two components, one of which, say A, contains y. By symmetry, $-A$ must be the other component. As $-y \in -A$, any segment from y to $-y$ meets $\sigma_x \cup (-\sigma_x)$.

Looking now at the nondegenerate quadrilateral $xy(-x)(-y)$, we have $\rho(x, y) = \rho(-x, -y) \geq \rho(x, -x) = \rho(y, -y)$ and the diagonals intersect. This contradicts Lemma 3. □

In any Alexandrov space (with curvature bounded below) the metric is nonbifurcating [2]. For a definition of an *Alexandrov space* see [2]. A 2-dimensional Alexandrov space is called an *Alexandrov surface*.

COROLLARY 3. *If X is an Alexandrov surface homeomorphic to the 2-dimensional standard sphere then each diameter of X is antipodal.*

In the convex case Corollary 3 yields the following result of C. Vîlcu (Proposition 6 in [4]).

COROLLARY 4. *Each diameter of a compact convex surface in $\mathbb{R}^3$ symmetric with respect to the origin is antipodal.*

It is easily seen that the conclusion of Theorem 3 is also true in case X is convex in H. The next theorem generalizes this.

THEOREM 4. *If any two points of* X *are joined by a unique segment, then there exists an antipodal diameter of* X. *If, moreover,* ρ *is nonbifurcating, then each diameter of* X *is antipodal.*

Proof. Put $\operatorname{diam} X = 1$. Let again $x, y \in X$ verify $y \neq -x$ and $\rho(x, y) = \operatorname{diam} X$. Let σ_{xy} be the segment from x to y, and σ_x the segment from x to $-x$. We have $\mathbf{0} \in \sigma_x$; indeed, if σ_x is parametrized by arc-length through $f : [0, 1] \to \sigma_x$, since $-f(t) = f(1 - t)$, we have $-f(1/2) = f(1/2) = \mathbf{0}$. Moreover, $\mathbf{0}$ is the midpoint of σ_x.

Similarly, $\mathbf{0}$ is the midpoint of the segment σ_y joining y to $-y$. As $\rho(x, \mathbf{0}) + \rho(\mathbf{0}, y) \geq \rho(x, y)$, we have

$$\rho(x, -x) + \rho(y, -y) \geq 2,$$

which implies $\rho(x, -x) = \rho(y, -y) = 1$. This proves the first part of the theorem.

For the second part, consider the quadrilateral $xy(-x)(-y)$, observe that $\rho(x, y) = \rho(-x, -y) = 1$ and $\sigma_x \cap \sigma_y = \{\mathbf{0}\}$, and apply Lemma 3. □

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