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Minkowski's theorem for arbitrary convex sets

Tudor Zamfirescu[1]

Department of Mathematics, University of Dortmund, 44221 Dortmund, Germany

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Abstract

V. Klee extended a well-known theorem of Minkowski to non-compact convex sets. We generalize Minkowski's theorem to convex sets which are not necessarily closed.

By a classical theorem of Minkowski, every compact convex set is the convex hull of its set of extreme points.

Our goal is to find a variant of Minkowski's theorem having larger applicability than the original version. Already Klee found such a variant allowing unbounded closed convex sets [2]. We shall extend the family of convex sets by relaxing the requirement that they are closed.

Let A be a convex set in $\mathbb{R}^d$ with non-empty interior.

We first reproduce from [3] the definition of a face. A convex subset $F \subset A$ is called a *face of* A if $x, y \in A$ and $(x + y)/2 \in F$ imply $x, y \in F$.

Thus, an *extreme point of* A is a face consisting of a single point. Similarly, an *extreme ray* of A is a closed half-line which is a face of A. And A itself is also a face. Every point of A lies in the interior $\operatorname{int} A$ of A or in a face distinct from A. Let $\operatorname{ext} A$, $\operatorname{extr} A$, and $\overline{A}$ denote the set of all extreme points, the union of all extreme rays, and the closure of A, respectively.

If $x \in A$, the intersection of A with a closed half-space containing x is called a *cap of* x. For $V \subset \mathbb{R}^d$, $\operatorname{conv} V$ denotes the convex hull of V.

The following well-known result is due essentially to Steinitz [4] (see also [1]).

Lemma 1. *If the convex set* A *is compact, any interior point of* A *belongs to the interior of the convex hull of at most* $2d$ *extreme points of* A.

E-mail address: tudor.zamfirescu@mathematik.uni-dortmund.de.

[1] Present address: Institute of Mathematics of the Romanian Academy, Bucharest 14700, Romania.

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We shall also make use of the following result.

Lemma 2 ([3]). *A point $x \in A$ is an extreme point of A if and only if there exist caps of x of arbitrarily small diameter.*

Theorem 1. *Let A be convex and bounded. Then* $\operatorname{conv} \operatorname{ext} A = A$ *if and only if, for every face F of A,* $\operatorname{ext} F$ *is dense in* $\operatorname{ext} \overline{F}$.

This theorem strengthens Minkowski's theorem because, if A is compact, every face F of A is closed, and $\operatorname{ext} F = \operatorname{ext} \overline{F}$.

Proof. First, suppose $\operatorname{ext} F$ is dense in $\operatorname{ext} \overline{F}$ for every face F of A. We show that $\operatorname{conv} \operatorname{ext} A = A$.

We use induction on the dimension d of A.

The statement is obvious for dimension 1: in this case A must be a closed line-segment, and the conclusion is true.

Suppose $\dim A = d$ and assume the statement true for all dimensions less than d.

Consider any face F of A different from A. Then, by the induction hypothesis, $\operatorname{conv} \operatorname{ext} F = F$. Since $\operatorname{ext} F \subset \operatorname{ext} A$, $F \subset \operatorname{conv} \operatorname{ext} A$.

It remains to show that $\operatorname{int} A \subset \operatorname{conv} \operatorname{ext} A$.

Let $a \in \operatorname{int} A$. We find finitely many (in fact we don't need more than $2d$ by Lemma 1) points $a_1, a_2, \ldots, a_k$ in $\operatorname{ext} \overline{A}$ such that

$$a \in \operatorname{intconv}\{a_1, a_2, \ldots, a_k\}.$$

Now, clearly, if $a_i' \in \operatorname{ext} A$ is chosen close enough to a_i $(i = 1, \ldots, k)$, then

$$a \in \operatorname{intconv}\{a_1', a_2', \ldots, a_k'\} \subset \operatorname{conv} \operatorname{ext} A.$$

Assume now that $\operatorname{ext} F$ is not dense in $\operatorname{ext} \overline{F}$ for some face F. Then there is an extreme point e of $\overline{F}$ and a neighbourhood N of e in $\operatorname{aff} F$ such that $N \cap \operatorname{ext} F = \emptyset$. By Lemma 2, there exists a cap of e included in N. This prevents e from being in $\operatorname{conv} \operatorname{ext} F$. Lying in $\operatorname{aff} F$, $e \notin \operatorname{conv}(A \setminus F)$ either. Hence $\operatorname{conv} \operatorname{ext} A \neq A$. □

Klee's generalization [2] of Minkowski's theorem can be strengthened in the same way.

Theorem 2. *Let A be convex and line-free. For every face F of A assume* $\operatorname{ext} F \cup \operatorname{extr} F$ *is dense in* $\operatorname{ext} \overline{F} \cup \operatorname{extr} \overline{F}$. *Then* $\operatorname{conv}(\operatorname{ext} A \cup \operatorname{extr} A) = A$.

We shall prove here a slightly stronger statement.

Theorem 3. *Let A be convex and line-free. Suppose for each face F of A,* $\operatorname{ext} F$ *is dense in* $\operatorname{ext} \overline{F}$ *and on each extreme ray R of A we can choose a (possibly empty) set $M(R)$ such that the intersection of R with $\overline{\cup_R M(R)}$ is unbounded. Then* $\operatorname{conv}(\operatorname{ext} A \cup (\cup_R M(R))) = A$.

Proof. We use again induction on the dimension d of A. The statement is obvious for $d = 1$.

Suppose $\dim A = d$ and assume the statement true for all A of smaller dimension. Consider any face F of A different from A. Then, by the induction hypothesis,

$$F = \operatorname{conv}(\operatorname{ext} F \cup (\cup_{R \subset F} M(R))) \subset \operatorname{conv}(\operatorname{ext} A \cup (\cup_{R \subset A} M(R))).$$

It remains to show that

$$\operatorname{int} A \subset \operatorname{conv}(\operatorname{ext} A \cup (\cup_{R \subset A} M(R))).$$

Let $a \in \operatorname{int} A$. Since A is line-free, there exists a cap C containing a in its interior. Then apply Lemma 1 to a and $\overline{C}$. Since every point of $\operatorname{ext} \overline{C} \setminus \operatorname{ext} \overline{A}$ lies in a line-segment between two extreme points of $\overline{A}$ or in $\operatorname{extr} \overline{A}$, there exist finitely many points $a_1, \ldots, a_k \in \operatorname{ext} \overline{A}$ and $b_1, \ldots, b_j \in \operatorname{extr} \overline{A}$, such that $a \in \operatorname{int} \operatorname{conv} \{a_1, \ldots, a_k, b_1, \ldots, b_j\}$. Now, each b_i lies on a segment $c_i d_i$ included in an extreme ray R of $\overline{A}$, with $c_i \in \operatorname{ext} \overline{A}$ and $d_i \in \overline{\cup_R M(R)}$.

Hence

$$a \in \operatorname{intconv}\{a_1, \ldots, a_k, c_1, \ldots, c_j, d_1, \ldots, d_j\}.$$

If $a_i' \in \operatorname{ext} A (1 \le i \le k)$, $c_i' \in \operatorname{ext} A$ $(1 \le i \le j)$, and $d_i' \in \cup_R M(R)$ $(1 \le i \le j)$ are chosen close enough to a_i, c_i, d_i respectively, then

$$a \in \operatorname{intconv}\{a_1', \ldots, a_k', c_1', \ldots, c_j', d_1', \ldots, d_j'\} \subset \operatorname{conv}(\operatorname{ext} A \cup (\cup_R M(R))). \quad \square$$

The following example illustrates the larger applicability of Theorem 1, compared with Minkowski's theorem.

Example. Let the unit circle C be written in polar coordinates $(\phi, 1)$, and take off the arcs

$$\frac{1}{3}\pi \le \phi \le \frac{2}{3}\pi, \qquad \frac{1}{3} \cdot \frac{1}{3}\pi \le \phi \le \frac{2}{3} \cdot \frac{1}{3}\pi, \qquad \frac{2}{3}\pi + \frac{1}{3} \cdot \frac{1}{3}\pi \le \phi \le \frac{2}{3}\pi + \frac{2}{3} \cdot \frac{1}{3}\pi,$$

etc. Let B be the convex hull of the remaining set, placed in the $x\mathbf{0}y$ plane of $\mathbb{R}^3$, with the centre of C at the origin $\mathbf{0}$ of $\mathbb{R}^3$, and put

$$A = \{(x, y, z) : z < 0 \wedge x^2 + y^2 + z^2 \le 4\}.$$

Then A, B, and $A \cup B$ are examples of non-closed convex sets to which Theorem 1 can be applied.

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