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Hamiltonian Properties of Generalized Halin Graphs

Dedicated to Ted Bisztriczky, on his sixtieth birthday.

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Abstract. A Halin graph is a graph $H = T \cup C$, where T is a tree with no vertex of degree two, and C is a cycle connecting the end-vertices of T in the cyclic order determined by a plane embedding of T. In this paper, we define classes of generalized Halin graphs, called k-Halin graphs, and investigate their Hamiltonian properties.

1 Introduction

A *Halin graph* is a graph H which is the union of a tree $T \neq K_2$ with no vertex of degree 2 and a cycle C connecting the end-vertices of T in the cyclic order determined by a plane embedding of T. Halin graphs are planar, 3-connected, and possess rather strong Hamiltonian properties. They are 1-*Hamiltonian*, *i.e.,* they are Hamiltonian [2] and remain so after the removal of any single vertex, as Bondy showed (see [4]). Moreover, Barefoot proved that they are *Hamiltonian connected*, *i.e.,* they admit a Hamiltonian path between every pair of vertices [1]. Bondy and Lovász [3] and, independently, Skowrońska [6] proved that Halin graphs on n vertices are almost pancyclic; more precisely, they contain cycles of all lengths l ($3 \leq l \leq n$) except possibly for a single even length. Also, they showed that Halin graphs on n vertices whose vertices of degree 3 are all on the outer cycle C must be *pancyclic*, *i.e.,* they must contain cycles of all lengths from 3 to n, thus proving a conjecture of Malkevitch [5]. Can we generalize the notion of a Halin graph such that some of its Hamiltonian properties are preserved? In the present paper this is what we do. We generalize Halin graphs in the following way.

A 2-connected planar graph G without vertices of degree 2, possessing a cycle C such that

(i) all vertices of C have degree 3 in G,
(ii) $G - C$ is connected and has at most k cycles

is called a *k-Halin graph*. The cycle C is called the *outer cycle* of G. The vertices and cycles in $G - C$ are called *inner vertices* and, respectively, *inner cycles* of G.

A 0-*Halin* graph is a usual *Halin* graph. Moreover, the class of k-Halin graphs is contained in the class of $(k+1)$-Halin graphs ($k \geq 0$). Thus we get a nested sequence of generalized Halin graphs. We shall see that, as expected, the Hamiltonicity

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of k-Halin graphs steadily decreases as k increases. Indeed, a 1-Halin graph is still Hamiltonian, but not necessarily Hamiltonian connected, a 2-Halin graph is not always Hamiltonian but still traceable, while a 3-Halin graph is not even necessarily traceable. The property of being 1-Hamiltonian, Hamiltonian connected or almost pancyclic is not preserved, even by 1-Halin graphs. However, Bondy and Lovász' result about the pancyclicity of Halin graphs with no inner vertex of degree 3 remains true even for 3-Halin graphs.

2 Hamiltonicity of 3-Halin Graphs

The graph obtained from a Halin graph by deleting a vertex x from its outer cycle is called a *reduced Halin graph* [3]. The three neighbouring vertices of x, whose degrees reduce by one, are called the *end-vertices* of the reduced Halin graph. Lemma 1 of [3] tells us the following.

Lemma 2.1 *In any reduced Halin graph each pair of end-vertices is joined by a Hamiltonian path.*

Lemma 2.1 will allow us to contract any reduced Halin subgraph of a graph G to a single vertex of degree 3, whenever we study Hamiltonian properties of G.

A path in a k-Halin graph will be called an *inner path*, if it has its end-vertices on distinct inner cycles and no other vertex on any inner or outer cycle.

We call a k-Halin graph ($k \geq 1$) *simple* if it is spanned by the union of all its inner paths and cycles and the outer cycle. Thus, a 1-Halin graph is simple if it has an inner cycle C_1 (besides the outer cycle C), and is spanned by $C \cup C_1$.

Theorem 2.2 *Every* 1*-Halin graph is Hamiltonian.*

Proof If the 1-Halin graph is also Halin, then it is Hamiltonian. Now let G be a 1-Halin graph with C_1 and C as its inner and outer cycles respectively (see Figure 1).

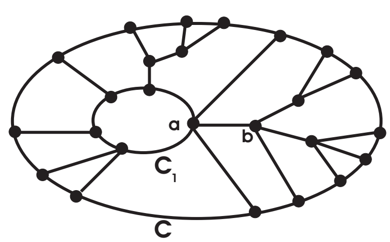

Figure 1

Let a be a vertex on C_1 and let $b \notin C_1$ be a neighbour of a. If $b \notin C$, the union of all paths from b to C, which do not contain a, is a tree T_b. This tree plus the edges on C between its leaves defines a reduced Halin graph H_b. We replace H_b by a single vertex $b' \in C$, adjacent with $a \in C_1$. If $b \in C$, we keep the edge ab. After doing this with all vertices of C_1, G reduces to a simple 1-Halin graph consisting of the two cycles C and C_1, and of edges between the two cycles, such that the outer cycle has only vertices of degree 3 (see Figure 2). A Hamiltonian cycle in this graph is shown in Figure 2. ■

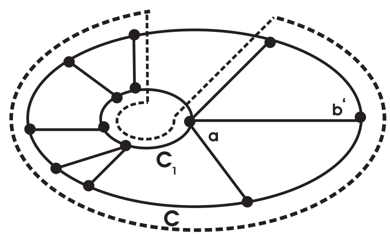

Figure 2

Remark. A 1-Halin graph is not necessarily Hamiltonian connected. Indeed, Figure 3 shows a bipartite 1-Halin graph G with 4 black and 4 white vertices. A path between any pair of white vertices will have one more white vertex than black, so it cannot be Hamiltonian.

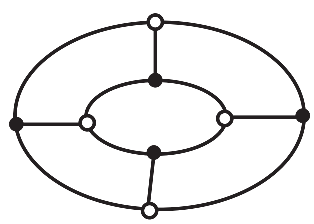

Figure 3: 1-Halin graph

A 2-Halin graph is not necessarily Hamiltonian. Indeed, Figure 4 shows a bipartite 2-Halin graph on 15 vertices. Such a graph has no Hamiltonian cycle.

Recall that a graph admitting a spanning path is called *traceable*, and the path is called *Hamiltonian*.

Theorem 2.3 *Every* 2*-Halin graph is traceable.*

Proof If the 2-Halin graph is also 1-Halin, then, by Theorem 2.2, it is Hamiltonian, hence traceable. Now let G be a 2-Halin graph with inner cycles C_1 and C_2 and outer cycle C, as shown in Figure 5.

Lemma 2.1 allows us to reduce G to a simple 2-Halin graph, that is, the union of C, C_1, C_2, and the unique path P between C_1 and C_2 in $G - C$ (possibly reduced to a vertex), plus edges between C and $C_1 \cup C_2 \cup P$ (see Figure 6). Let $a_1 \in C_1$

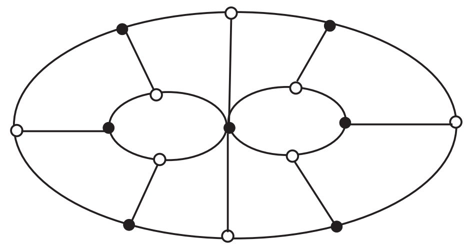

Figure 4: 2-Halin graph

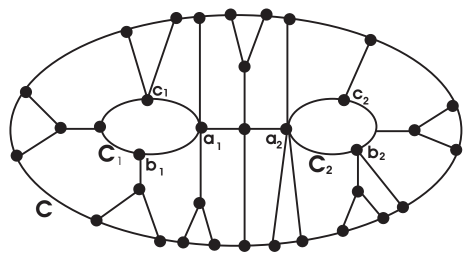

Figure 5

and $a_2 \in C_2$ be the endpoints of P. We claim that there is a Hamiltonian path in G between the neighbour b_1 or c_1 of a_1 on C_1 and the neighbour b_2 or c_2 of a_2 on C_2. This is illustrated in Figure 6, where a path between b_1 and b_2 is realized. Accordingly, G is traceable. ■

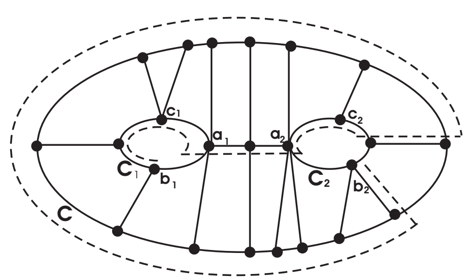

Figure 6

Remark. A 3-Halin graph is not necessarily traceable. Indeed, Figure 7 shows a 3-Halin bipartite graph G with 22 vertices coloured in two colours, 12 black and 10 white.

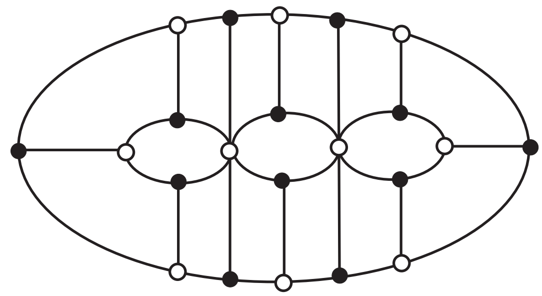

Figure 7: 3-Halin graph

3 Pancyclicity of 3-Halin Graphs

A graph on n vertices is called *almost pancyclic*, if it contains cycles of all lengths from 3 to n except possibly for one single length. Let us call *m-almost pancyclic* an almost pancyclic graph without cycles of length m.

As announced in the Introduction, we show here that all 3-Halin graphs without inner vertices of degree 3 are pancyclic, thus extending the corresponding result of Bondy and Lovász [3] on Halin graphs. We shall make use of the following central result of [3].

Lemma 3.1 *Every Halin graph is almost pancyclic. If the Halin graph H is m-almost pancyclic, then m is even and H must contain one of the three types of subgraphs depicted in Figure* 8.

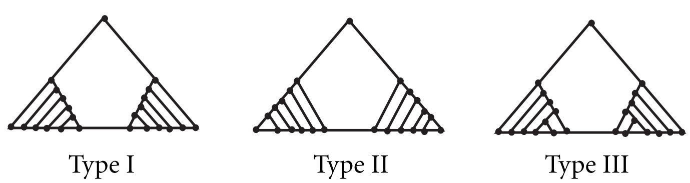

Figure 8: $(m = 12)$

Theorem 3.2 *Every* 3*-Halin graph without inner vertices of degree* 3 *is pancyclic.*

Proof Let G be a 3-Halin graph without inner vertices of degree 3. There are at most 3 inner cycles in G. Choose an edge in each of them, such that no pair of edges has a common point. The total number of chosen edges can be two if the union of the 3 inner cycles is a cycle plus a chord. Delete these chosen edges. The resulting Halin graph H has at most 6 inner vertices of degree 3.

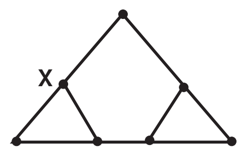

Figure 9

By Lemma 3.1, H is almost pancyclic. Assume cycles of length m are missing. Then, by Lemma 3.1, m is even and H must contain a reduced Halin graph of one of the types I, II, or III (Figure 8).

Suppose first that $m = 4$. Then H must contain a reduced Halin graph H' as described in Figure 9.

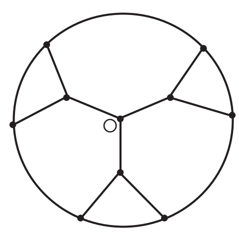

Figure 10

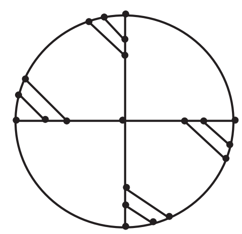

Figure 11

The point x of H' has degree 3. Hence it must belong in G to an edge e which has been deleted to obtain H. If the other endpoint of e is a vertex like x, *i.e.,* a non-endpoint of a subgraph of H isomorphic to H', then G has a cycle of length 4, and is therefore pancyclic. So, assume that the other endpoint of e is not a vertex like x. Since there are at most 3 edges like e, there are at most 3 vertices like x. But 4-almost pancyclic Halin graphs (see Figure 10) have more than 3 vertices like x if they are different from the graph H'' of Figure 10. In case $H = H''$, the vertex o must, on one hand, have degree at least 4 in G, but can, on the other hand, be no endpoint of any further edge of G. Thus, in any case we obtain a contradiction.

Suppose now that $m = 6$. The smallest 6-almost pancyclic Halin graph is shown in Figure 11. This graph has 8 inner vertices of degree 3, so it cannot be H.

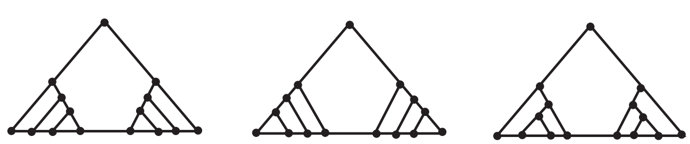

Figure 12

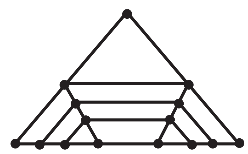

Figure 13

If $m = 8$, then, by Lemma 3.1, H must contain one of the reduced Halin subgraphs of Figure 12. Thus H has at least 6 inner vertices of degree 3, but they cannot be endpoints of only 3 edges in G, excepting the cases shown in Figure 13. In these cases, however, G has cycles of length 8, and is therefore pancyclic.

If $m \geq 10$, then the reduced Halin graph which must, by Lemma 3.1, appear in H has at least 8 inner vertices of degree 3, which is impossible. ■

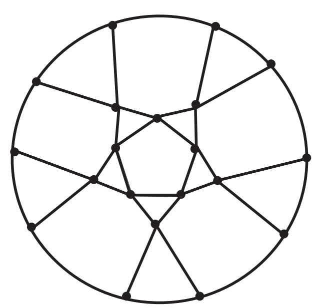

Figure 14

The 37-Halin graph of Figure 14 has no cycle of length 4 and shows that not every k-Halin graph with no inner vertex of degree 3 must be pancyclic. So we are led to the following question.

Which is the maximal number k for which every k-Halin graph with no inner vertex of degree 3 is pancyclic?

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