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Hamiltonian Connectedness in Directed Toeplitz Graphs

by
Shabnam Malik and Tudor Zamfirescu

Abstract

A directed Toeplitz graph is a digraph with a Toeplitz adjacency matrix. In this paper we investigate the hamiltonian connectedness of directed Toeplitz graphs.

Key Words: Toeplitz graph; Hamiltonian graph; Hamiltonian connected; Weakly hamiltonian connected.
2000 Mathematics Subject Classification: Primary 05C20, Secondary 05C45.

1 Introduction

A *Toeplitz matrix* is a square matrix (a_{ij}) which has constant values along all diagonals parallel to the main diagonal, i.e. , $a_{i,j} = a_{i+1,j+1}$. A *directed Toeplitz graph* is, by definition, a directed graph without loops with a Toeplitz adjacency matrix.

The main diagonal of this $(n \times n)$ Toeplitz adjacency matrix will be labeled 0 and it contains only zeros. The $n-1$ distinct diagonals above the main diagonal will be labeled $1, 2, \ldots, n-1$ and those under the main diagonal will also be labeled 1, 2, …, $n-1$. Let $s_1, s_2, \ldots, s_k$ be the upper diagonals containing ones and $t_1, t_2, \ldots, t_l$ be the lower diagonals containing ones, such that $0 < s_1 < s_2 < \cdots < s_k < n$ and $0 < t_1 < t_2 < \cdots < t_l < n$. Then, the corresponding Toeplitz graph will be denoted by $T_n\langle s_1, s_2, \ldots, s_k; t_1, t_2, \ldots, t_l\rangle$. Hence $T_n\langle s_1, s_2, \ldots, s_k; t_1, t_2, \ldots, t_l\rangle$ is the graph with vertices 1, 2, …, n, in which the edge (i, j) occurs if and only if $j-i = s_p$ or $i-j = t_q$ for some p and q ($1 \le p \le k$, $1 \le q \le l$).

Remark that $T_n\langle s_1, \ldots, s_i; t_1, \ldots, t_j\rangle$ and $T_n\langle t_1, \ldots, t_j; s_1, \ldots, s_i\rangle$ are obtained from each other by reversing the orientation of all edges. If the adjacency matrix is symmetric, the graph is said, as usual, to be undirected. $T_n\langle t_1, t_2, \ldots, t_i\rangle$ denotes the undirected Toeplitz graph with the adjacency matrix of $T_n\langle t_1, \ldots, t_i; t_1, \ldots, t_i\rangle$.

A directed graph G is *hamiltonian connected* if for any pair of distinct vertices a and b of G, there exists a hamiltonian path from a to b. G is *weakly hamiltonian connected* if for any pair of distinct vertices a and b of G, there exists a hamiltonian path from a to b or one from b to a.

After the study of circulant graphs, in recent years the more general undirected Toeplitz graphs have been investigated. They were introduced in [1]. Properties of these graphs, such as bipartiteness, planarity and colourability, have been studied in [2], [3] and [4]. Hamiltonian properties have been investigated by R. van Dal et al [1] and C. Heuberger [5], while the directed case was studied by S. Malik and A.M. Qureshi [6].

In this paper we start the study of directed Toeplitz graphs with respect to the much stronger property of being hamiltonian connected.

From now on by a Toeplitz graph we shall always mean a directed one, the undirected case being specified as a particular case when needed. We start our investigation with small values of k and l. It is immediately seen that $T_n\langle 1,2\rangle$ is hamiltonian connected only for $n = 3$. Thus, it is natural to ask what happens if a little larger Toeplitz graphs, keeping $s_2 = t_2 = 2$, are considered. We shall see that we are quickly successful. One more diagonal of 1's already leads to weak hamiltonian connectedness. And two such diagonals, one above the main diagonal and the other below guarantee hamiltonian connectedness.

The case $s_2 = t_2 = 2$ will be completely treated. There are several other classes of Toeplitz graphs with small k and l, which should also be treated. In this paper we choose to study only one more situation, in which we can prove hamiltonian connectedness: $T_n\langle 1,2,3;2,3\rangle$.

2 Toeplitz Graph with $s_2 = t_2 = 2$ and $k = 3$

In this section we will show that most Toeplitz graphs on n vertices strictly including $T_n\langle 1,2;1,2\rangle$ are hamiltonian connected for all n.

Theorem 1. *$T_n\langle 1,2,s;1,2\rangle$ is weakly hamiltonian connected for all s, and hamiltonian connected for $s \geq 5$.*

Proof: First we remark that, for any vertex $a \neq 1$ of $T_n\langle 1,2,s;1,2\rangle$, there exists a path La from a to $a-1$ containing all vertices in $\{1, 2, \ldots, a\}$, namely

$$(a,\ a-2,\ a-4,\ \ldots,\ 4,\ 2,\ 1,\ 3,\ 5,\ \ldots,\ a-3,\ a-1)$$

or

$$(a,\ a-2,\ a-4,\ \ldots,\ 5,\ 3,\ 1,\ 2,\ 4,\ \ldots,\ a-3,\ a-1),$$

depending upon the parity of a (see Fig 1). For $a = 1$, we put $L1 = 1$.
Similarly for any $a \neq n$ there is a path Ra from $a+1$ to a containing all vertices in $\{a,\ a+1,\ \ldots,\ n-1,\ n\}$ (see Fig 2). For $a = n$, we put $Rn = n$.

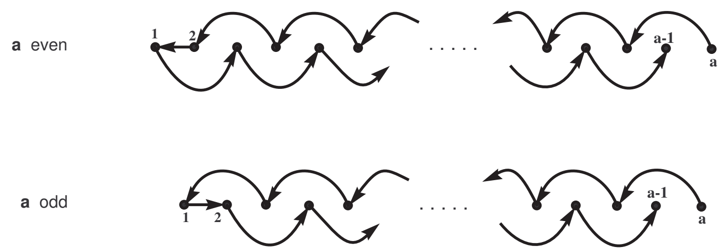

Figure 1: La

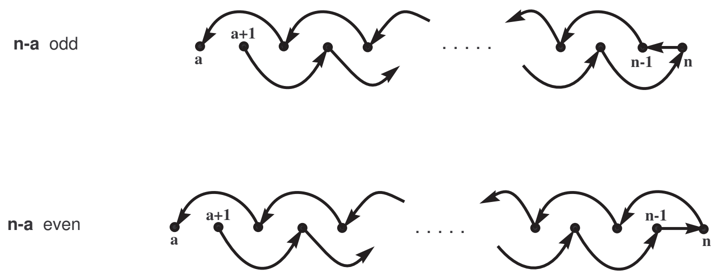

Figure 2: Ra

Now let a and b be distinct vertices of the Toeplitz graph $T_n\langle 1, 2, s; 1, 2\rangle$. We assume that $a < b$. We show that there exists a hamiltonian path from a to b. Two cases arise.

Case 1. $b \neq a + 1$.

In this case a hamiltonian path from a to b is $(La,\ a+1,\ a+2,\ \ldots,\ b-1,\ Rb)$ (see Fig 3).

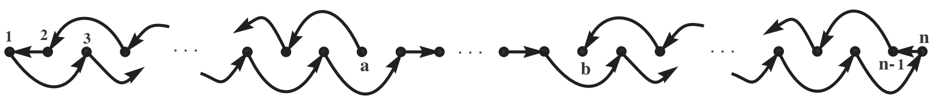

Figure 3: $T_n\langle 1, 2, s; 1, 2\rangle$, where $b \neq a + 1$

Case 2. $b = a + 1$.

In this case, assume first that $s \geq b$. Then a hamiltonian path from a to b is $(a,\ a-1,\ a-2,\ \ldots,\ 1,\ Rs,\ s-1,\ s-2,\ \ldots,\ b)$ (see Fig 4).

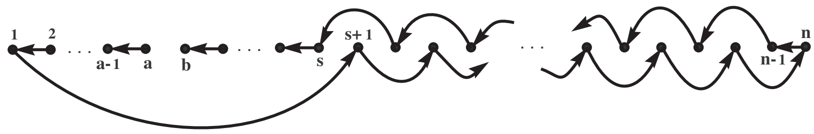

Figure 4: $T_n\langle 1, 2, s; 1, 2\rangle$, where $b = a + 1$ and $s \geq b$

Suppose now that $s < b < n$. Then a hamiltonian path from a to b is $(a,\ a-1,\ a-2,\ \ldots,\ a-s+4,\ L(a-s+3),\ Rb)$ (see Fig 5).

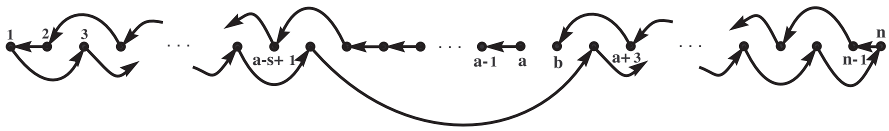

Figure 5: $T_n\langle 1, 2, s; 1, 2\rangle$, where $b = a + 1$ and $s < b < n$

There is only one subcase left, namely $s < b = n$. Then a hamiltonian path from a to b is $(a,\ a-1,\ a-2,\ \ldots,\ n-s+2,\ L(n-s+1),\ b)$ (see Fig 6).

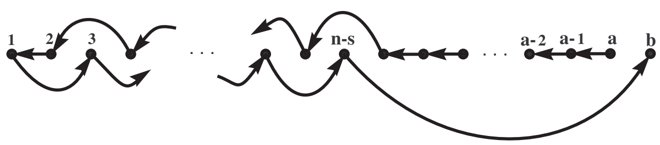

Figure 6: $T_n\langle 1, 2, s; 1, 2\rangle$, where $b = a + 1$ and $s < b = n$

We prove now the existence of a hamiltonian path from b to a in $T_n\langle 1, 2, s; 1, 2\rangle$, for $s \geq 5$.

Again two cases arise.

Case 1. $b \neq a + 1$.

By reversing the direction of edges in the previous Case 1, we will get a hamiltonian path from b to a.

Case 2. $b = a + 1$

In this case, assume first that $s > b$. Then a hamiltonian path from b to a is $(b,\ a-1,\ a-2,\ a-3,\ \ldots,\ 1,\ Rs,\ s-1,\ s-2,\ s-3,\ \ldots,\ a+2,\ a)$, see Fig 7.

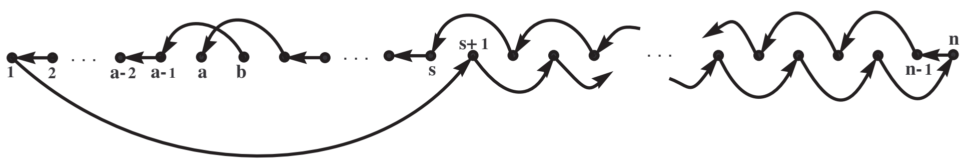

Figure 7: $T_n\langle 1, 2, s; 1, 2\rangle$, where $b = a + 1$ and $s > b$

Suppose now that $s \leq b < n - 1$. Then a hamiltonian path from b to a is $(b,\ a-1,\ a-2,\ a-3,\ \ldots,\ b-s+4,\ L(b-s+3),\ R(b+1),\ a)$, see Fig 8.

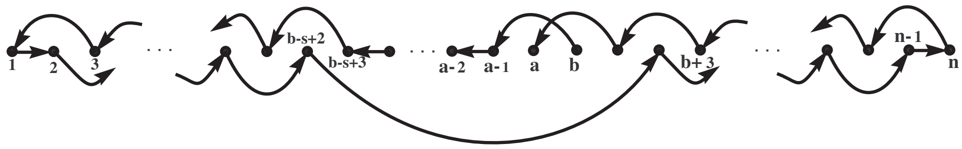

Figure 8: $T_n\langle 1, 2, s; 1, 2\rangle$, where $b = a + 1$ and $s \leq b < n - 1$

Suppose that $s \leq b = n - 1$. Then a hamiltonian path from b to a is $(b,\ a-1,\ a-2,\ a-3,\ \ldots,\ n-s+2,\ L(n-s+1),\ n,\ a)$, see Fig 9.

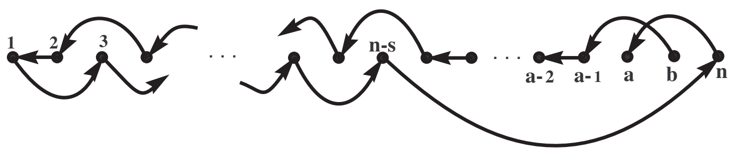

Figure 9: $T_n\langle 1, 2, s; 1, 2\rangle$, where $b = a + 1$ and $s \leq b = n - 1$

The only subcase which remains is $s \leq b = n$. In this case, Ln is a hamiltonian path from b to a.

This finishes the proof. $\square$

By Theorem 1, $T_n\langle 1,2,s;1,2\rangle$ is weakly hamiltonian connected if $3 \le s \le 4$. Can this statement be strengthened for some n's?

Theorem 2. *Let $3 \le s \le 4$. The Toeplitz graph $T_n\langle 1,2,s;1,2\rangle$ is hamiltonian connected if and only if $n = s+1$.*

Proof: Assume $T_n\langle 1,2,3;1,2\rangle$ is hamiltonian connected. Then there is a hamiltonian path H from vertex 3 to vertex 2. Starting from vertex 3, the only way to reach vertex 1 is using the edge (3, 1) because (2, 1) is not on H. Then, the edges (1, 4) and (4, 2) necessarily belong to H. Thus, $5 \notin H$ and therefore $n = 4$. For $s = 4$, there is a hamiltonian path from vertex 4 to vertex 3 and we similarly see that it can only be (4, 2, 1, 5, 3). Hence $n = 5$.

Conversely, one can easily verify that $T_4\langle 1,2,3;1,2\rangle$ and $T_5\langle$ 1, 2, 4; 1, 2$\rangle$ are hamiltonian connected.

This finishes the proof. □

Thus by Theorems 1 and 2, $T_n\langle 1,2,n-1;1,2\rangle$ is hamiltonian connected for all n.

Theorem 3. $T_n\langle 1,2,s;1,2,t\rangle$ *is hamiltonian connected for all n, s and t.*

Proof: In the proof of Theorem 1 it is shown that, for $a < b$, there always exists a hamiltonian path from a to b in $T_n\langle 1,2,s;1,2\rangle$. Reversing all the edges of $T_n\langle 1,2,t;1,2\rangle$ we obtain a hamiltonian path from b to a in $T_n\langle 1,2;1,2,t\rangle$. Since

$$T_n\langle 1,2,s;1,2,t\rangle = T_n\langle 1,2,s;1,2\rangle \cup T_n\langle 1,2;1,2,t\rangle,$$

the hamiltonian connectedness of this graph follows. □

Theorem 3 is of course relevant only for $s, t \in \{3, 4\}$, otherwise it follows directly from Theorem 1.

Corollary 1. $T_n\langle 1,2,s\rangle$ *is hamiltonian connected for all n and s.* □

3 Toeplitz Graph with $t_1 = 2$ and $s_3 = t_2 = 3$

Theorem 4. $T_n\langle 1, 2, 3; 2, 3\rangle$ *is hamiltonian connected for all* n.

Proof: First we remark that, for any vertex $a \neq 1$ of $T_n\langle 1, 2, 3; 2, 3\rangle$, there exists a path La from a to $a-1$ containing all vertices in $\{1, 2, \ldots, a\}$, namely

$$(a,\ a-2,\ a-4,\ \ldots,\ 4,\ 1,\ 2,\ 3,\ 5,\ 7,\ \ldots,\ a-1)$$

or

$$(a,\ a-2,\ a-4,\ \ldots,\ 5,\ 3,\ 1,\ 2,\ 4,\ \ldots,\ a-1),$$

depending upon the parity of a (see Fig 10). For $a = 1$, we put $L1 = 1$.

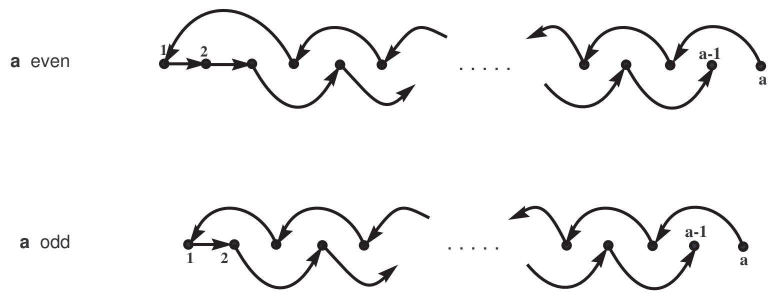

Figure 10: La

Similarly for any $a \neq n$ there is a path Ra from $a+1$ to a containing all vertices in $\{a, a+1, \ldots, n-1, n\}$, depending upon the parity of a and n, (see Fig 11). For $a = n$, we put $Rn = n$.

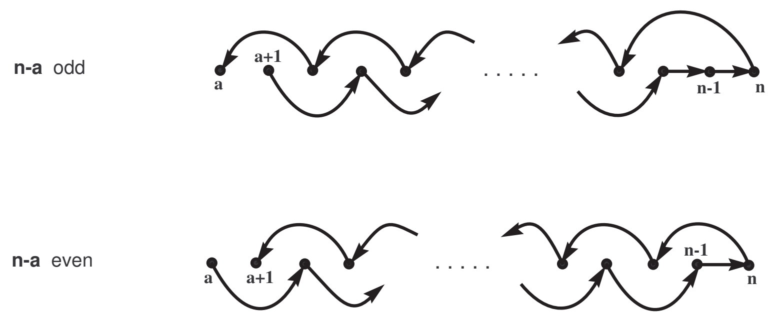

Figure 11: Ra

Now let a and b be distinct vertices of the Toeplitz graph $T_n\langle 1,2,3;2,3\rangle$. We assume that $a < b$. We show that there exists a hamiltonian path from a to b. Again, two cases arise.

Case 1. $b \neq a+1$.

In this cases a hamiltonian path from a to b is
$(La,\, a+1,\, a+2,\, \ldots,\, b-1,\, Rb)$, see Fig 12.

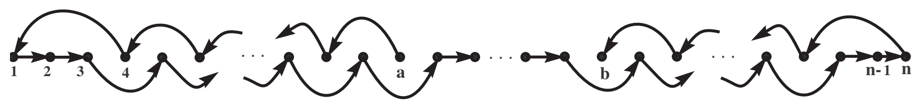

Figure 12: $T_n\langle 1,2,3;2,3\rangle$, where $b \neq a+1$

Case 2. $b = a+1$.

In this case a hamiltonian path from a to b is $(La,\, Rb)$, see Fig 13.

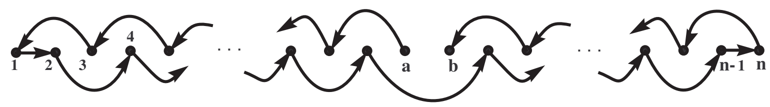

Figure 13: $T_n\langle 1,2,3;2,3\rangle$, where $b = a+1$

We prove now the existence of a hamiltonian path from b to a in $T_n\langle 1,2,3;2,3\rangle$. Another two cases arise.

Case 1. $b \neq a+1$.

1(i) Assume $b \notin \{3,\, n\}$. If a and b are of opposite parity then a hamiltonian path from b to a is $(b,\, R(b{+}1),\, b{-}2,\, b{-}1,\, b{-}4,\, b{-}3,\, \ldots,\, a{+}1,\, a{+}2,\, L(a{-}1),\, a)$, see Fig 14.

Figure 14: $T_n\langle 1,2,3;2,3\rangle$, where $b \neq a+1$ and a and b are of opposite parity

If a and b are of same parity then a hamiltonian path from b to a is (b, $R(b+1)$, $b-2$, $b-1$, $b-4$, $b-3$, $\ldots$, $a+1$, $a+2$, $a+3$, $L(a+1)$), see Fig 15.

Figure 15: $T_n\langle 1,2,3;2,3\rangle$, where $b \neq a+1$ and a and b are of same parity

1(ii) Suppose now $b = n$ and $a \neq n-2$. If a and b are of opposite parity then a hamiltonian path from b to a is (n, $n-2$, $n-1$, $n-4$, $n-3$, $\ldots$, $a+1$, $a+2$, $L(a-1)$, a), see Fig 16.

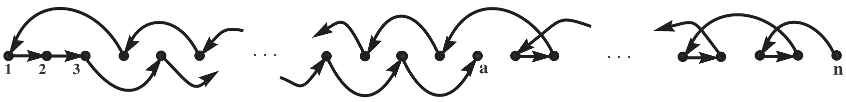

Figure 16: $T_n\langle 1,2,3;2,3\rangle$, where $b \neq a+1$, $b = n$, $a \neq n-2$ and a and b are of opposite parity

If a and b are of same parity then a hamiltonian path from b to a is (n, $n-2$, $n-1$, $n-4$, $n-3$, $\ldots$, $a+2$, $a+3$, $L(a+1)$), see Fig 17.

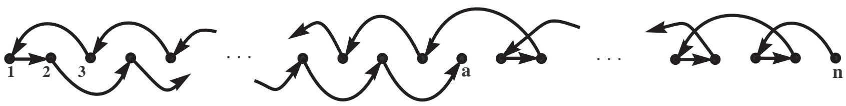

Figure 17: $T_n\langle 1,2,3;2,3\rangle$, where $b \neq a+1$, $b = n$, $a \neq n-2$ and a and b are of same parity

1(iii) Assume now $a = n - 2$. Then a hamiltonian path from b to a is (n, $n-3$, $n-1$, $L(n-4)$, $n-2$), see Fig 18.

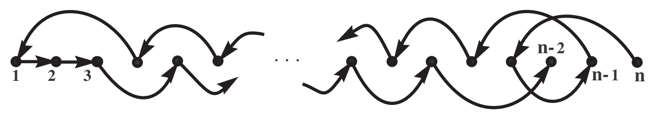

Figure 18: $T_n\langle 1,2,3;2,3\rangle$, where $b \neq a+1$ and $a = n-2$

1(iv) Finally, let $b = 3$. Then a hamiltonian path from b to a is (3, R5, 2, 4, 1), see Fig 19.

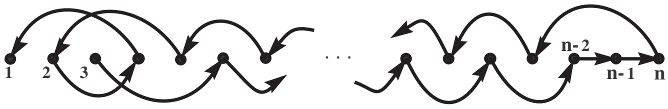

Figure 19: $T_n\langle 1,2,3;2,3\rangle$, where $b \neq a+1$ and $b = 3$

Case 2. $b = a+1$.

2(i) If $b \notin \{n-2, 4\}$ then a hamiltonian path from b to a is (b, $R(b+1)$, $L(a-1)$, a), see Fig 20.

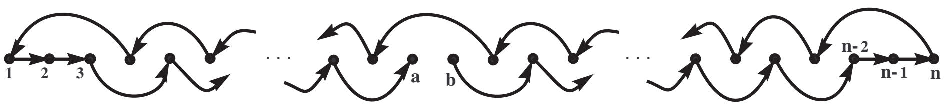

Figure 20: $T_n\langle 1,2,3;2,3\rangle$, where $b = a+1$ and $b \notin \{n-2, 4\}$

2(*ii*) If $b = n-2$ then a hamiltonian path from b to a is ($n-2$, $L(a-2)$, $a-1$, $n-1$, n, $n-3$), see Fig 21

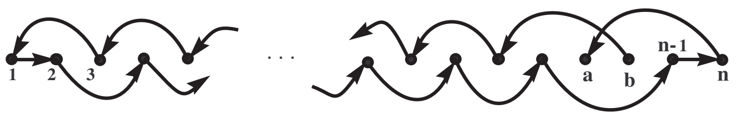

Figure 21: $T_n\langle 1,2,3;2,3\rangle$, where $b = a+1$ and $b = n-2$

2(*iii*) If $b = 4$ then a hamiltonian path from b to a is (4, 1, 2, 5, $R6$, 3), see Fig 22.

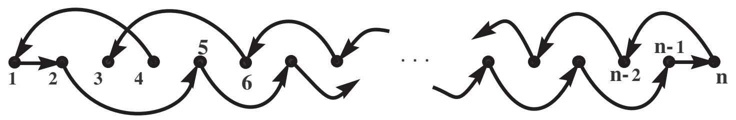

Figure 22: $T_n\langle 1,2,3;2,3\rangle$, where $b = a+1$ and $b = 4$

This finishes the proof. □

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Faculty of Mathematics,
Forman Christian College University,
Lahore, Pakistan.
E-mail: `shabnam.malik@gmail.com`

Faculty of Mathematics,
University of Dortmund,
Germany,
and
Institut of Mathematics of the Romanian Academy,
Bucharest, Romania.
E-mail: `tuzamfirescu@googlemail.com`