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NON-EXPANDING MAPPINGS IN GRAPHS

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Abstract

In this paper we introduce and start studying non-expanding mappings on graphs. We prove the invariance of certain types of subgraphs under such mappings. Then we focus our attention to contractions and λ-contractions as peculiar non-expanding mappings, and to trees and cacti as special kinds of graphs.

Introduction

Astonishingly, the topic of fixed points in graphs is rather little explored. It is, however, not inexistent. It appeared at least in connection with automorphisms in graphs. See, for example, the contributions of Imrich and Turner [2], Imrich, Kristic and Turner [1], and Seifter [3].

In this paper all graphs are finite, connected and non-directed, have no loops and no multiple edges.

A subgraph H of a graph G is called *invariant* (or *fixed*) under the mapping $f : V(G) \to V(G)$ if, for any point $v \in V(H)$, $f(v) \in V(H)$. (The elements of $V(G)$ are called *points* in this paper.) An edge $uv \in E(G)$ is said

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to be *inverted* under f if $f(u) = v$ and $f(v) = u$. A subgraph of G isomorphic to K_2 will also be called an *edge*. The usual metric in $V(G)$ will be denoted by d, i.e., $d(x, y)$ is the length of a shortest path from x to y. Such a shortest path is called a *segment*.

The mapping f is called *non-expanding* on G if, for any points $v, w \in V(G)$,

$$d(f(v), f(w)) \leq d(v, w),$$

as usual in any metric space. Let G be a graph and u, v be distinct points in $V(G)$. The union $S(u, v)$ of all segments from u to v will be called the *thick segment* from u to v.

Let n and h be positive integers, and let $a_1, a_2, \ldots, a_{n-1}, a_n = a_0$ be n distinct points in $V(G)$, such that $d(a_{i-1}, a_i) = h$ whenever $1 \leq i \leq n$. Then $\bigcup_{i=1}^{n} S(a_{i-1}, a_i)$ is called a *thick cycle* of G, with *vertices* at $a_1, \ldots, a_n$. (See Figure 1.) For example, all cycles and thick segments are thick cycles (the thick segments are precisely the thick cycles with two vertices). Note that $n \geq 2$, because $h \geq 1$.

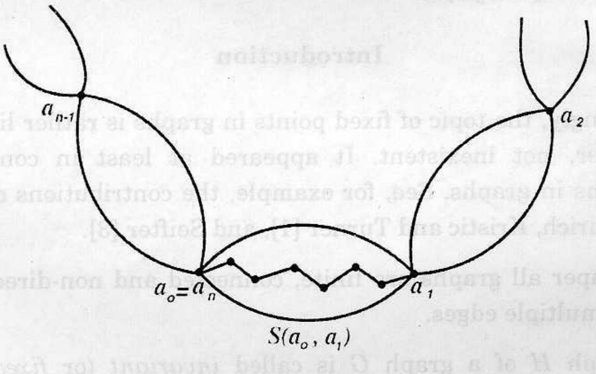

Figure 1

The extensive use of contractions and other non-expanding mappings in various metric spaces is notorious. On the other side, the standard approximation procedure is the use of discrete structures. Graphs, for example triangulations, are a most common kind of such a structure. This is the motivation for our interest in non-expanding mappings on graphs.

Non-Expanding Mappings

In this Section we present two basic results on non-expanding mappings in graphs without any further restrictions.

Theorem 1. *Each non-expanding mapping on a graph has a fixed point or an invariant thick cycle.*

Proof. Assume f is non-expanding on G and has no fixed point.

Take arbitrarily $a_0 \in V(G)$ and $a_1 = f(a_0)$. Similarly, let $a_i = f(a_{i-1})$ $(i = 2, 3, \ldots)$. The sequence $\{a_n\}_{n=1}^{\infty}$ cannot have all elements distinct, so $a_i = a_j$ for some indices $i < j$. We may assume that $i = 0$ and j is chosen to be minimal. Since f has no fixed points, $j > 1$. Because

$$d(a_0, a_1) \geq d(a_1, a_2) \geq \ldots \geq d(a_{j-1}, a_j) \geq d(a_0, a_1),$$

we have equality.

Consider a segment Σ_0 from a_0 to a_1. It has length h, say. Then all paths $\Sigma_1 = f(\Sigma_0)$, $\Sigma_2 = f(\Sigma_1)$, ... having length h, must be segments. Since $\Sigma_k \subset S(a_i, a_{i+1})$ for $i = k \bmod j$, the thick cycle with vertices at $a_1, a_2, \ldots, a_j$ is invariant under f.

If a thick cycle of G is not a path, we say that the thick cycle is *proper*.

Theorem 2. *Each non-expanding mapping on a graph either has a fixed point, or an inverted edge, or an invariant proper thick cycle.*

Proof. Assume f is non-expanding in G and has no fixed point.

By Theorem 1, there exists an invariant thick cycle C with vertices, say, $a_0, a_1, \ldots, a_{j-1}, a_j = a_0$.

If C is not proper, then it is a path P. The fact that $d(a_{i-1}, a_i)$ does not depend on i $(1 \leq i \leq j)$ and that $a_0, a_1, \ldots, a_{j-1}$ are distinct implies $j = 2$ and that a_0 and a_1 are the endpoints of P.

Let $P = v_0 v_1 \ldots v_{k-1} v_k$, where $v_0 = a_0$, $v_k = a_1$. Then necessarily $f(v_i) = v_{k-i}$ for all i.

If k is even, the midpoint $v_{k/2}$ is a fixed point, but we excluded this. Thus, k is odd, $f(v_m) = v_{m+1}$ and $f(v_{m+1}) = v_m$, where $m = \lfloor k/2 \rfloor$. This means that the edge $v_m v_{m+1}$ is inverted.

Of course, if G has less than three cycles, every thick cycle is a cycle, and f has a fixed point, or an inverted edge, or else an invariant cycle. It is interesting to determine circumstances under which G admits a fixed point, or an inverted edge, or an invariant cycle. Such a curcumstance is presented in the last Section. That this is not always the case can be seen from the example of $K_{2,3}$ with two white points w_1, w_2 and three black points b_1, b_2, b_3, on which the non-expanding mapping f is defined by $f(w_1) = w_2$, $f(w_2) = w_1$, $f(b_1) = b_2$, $f(b_2) = b_3$, $f(b_3) = b_1$. (See Figure 2.)

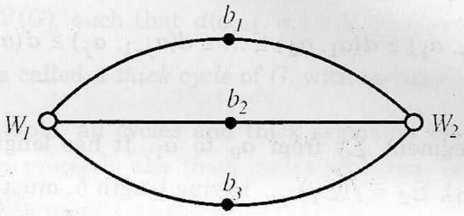

Figure 2

For $x \in V(G)$, we call, as usual, the sequence x, $f(x)$, $f^2(x)$, ... the *orbit* of x under f. We say that the orbit of x *enters* a subgraph $H \subset G$ if, for some m and all integers $n > m$, $f^n(x) \in V(H)$. It is also clear what should mean that an orbit enters a point or an edge.

Theorem 3. *Given a non-expanding mapping on a graph, any orbit either enters a fixed point, or an inverted edge, or else a proper thick cycle.*

Proof. This can be extracted from the proof of Theorem 2.

Theorems 4-7 in this paper also admit extensions stating that any orbit enters one of the small invariant subgraphs specified in the respective statements.

Let C be a thick cycle invariant under the non-expanding mapping f on G. Let $a_0, a_1, \ldots, a_{j-1}, a_j = a_0$ be the vertices of C, and let $\Sigma_i \in S(a_i, a_{i+1})$ for some $i \in \{0, 1, \ldots, j-1\}$. Put, as before, $\Sigma_{i+1} = f(\Sigma_i)$, and so on.

Since, for each positive integer n, $\Sigma_{nj} \in S(a_0, a_1)$ but there are only finitely many segments from a_0 to a_1, for some $m > n$ we must have $\Sigma_{mj} = \Sigma_{nj}$. Assume n and m are chosen to be minimal. Then $\bigcup_{i=nj}^{mj-1} \Sigma_i$ will be called a *kernel* of C.

A thick cycle may have more than one kernel, depending on the initial choice of Σ_i.

λ-Contractions

Let G be a graph, and let $0 < \lambda < 1$.

A mapping $f : V(G) \to V(G)$ is called a *λ-contraction* if, for any points $v, w \in V(G)$,

$$d(f(v), f(w)) \le \lceil \lambda d(v, w) \rceil.$$

Since every λ-contraction is a non-expanding mapping, it must admit a fixed point or an invariant thick cycle. The next result shows that, concerning the thick cycle, there is an upper bound on the diameter of any of its kernels.

Theorem 4. *Every λ-contraction on a graph has a fixed point or an invariant thick cycle, each kernel of which has diameter at most* $\lceil \lambda/(1-\lambda) \rceil$.

Proof. Assuming that f is a λ-contraction in G without fixed points, and proceeding like in the proof of Theorem 1, we obtain an invariant thick cycle C with vertices $a_1, \ldots, a_j$, and also the sequence of segments $\Sigma_i, \Sigma_{i+1}, \ldots$. Starting in this way, with Σ_i, we are led to a kernel B of C. With the numbers n, m from the definition of a kernel, for each $v \in V(B)$, we have $v \in V(\Sigma_{nj+k})$ for some $k \in \{0, 1, \ldots, mj-1\}$, and $f^{(m-n)j}(v) = v$. For any pair $v, w \in V(B)$, $f^{(m-n)j}(v) = v$ and $f^{(m-n)j}(w) = w$. Hence

$$d(v, w) \ge d(f(v), f(w)) \ge \ldots \ge d(f^{(m-n)j}(v), f^{(m-n)j}(w)) = d(v, w),$$

which yields equality among all terms. If $d(v, w) > \lceil \lambda/(1-\lambda) \rceil$, then

$$d(f(v), f(w)) \le \lceil \lambda d(v, w) \rceil < d(v, w),$$

because, for any positive integer s,

$$s > \left\lceil \frac{\lambda}{1-\lambda} \right\rceil \Leftrightarrow s \geq \frac{1}{1-\lambda} \Leftrightarrow \lceil \lambda s \rceil \leq s - 1.$$

Therefore $d(v, w) \leq \lceil \lambda/(1-\lambda) \rceil$ and the diameter of B is bounded above by $\lceil \lambda/(1-\lambda) \rceil$.

Corollary. *Each 1/2-contraction on a graph has a fixed point, or an inverted edge, or an invariant cycle. The invariant points, edges and cycles span, all together, a complete subgraph.*

Proof. Theorem 4 guarantees the existence part. Now any invariant thick cycle is a cycle, and any kernel of an invariant cycle is the cycle itself. Let now U be the union of all fixed points, inverted edges, and invariant cycles. Consider two distinct points $u, v \in V(U)$. For some integer n_u we have $f^{n_u}(u) = u$, and for some integer n_v, we have $f^{n_v}(v) = v$. Thus, also, $f^{n_u n_v}(u) = u$ and $f^{n_u n_v}(v) = v$. Then

$$d(u, v) \geq d(f(u), f(v)) \geq \ldots \geq d(f^{n_u n_v}(u), f^{n_u n_v}(v)) = d(u, v)$$

implies equality of all above terms, whence $d(u, v) = 1$.

Contractions

A non-expanding mapping $f : V(G) \to V(G)$ on the graph G is called a *contraction* if for any non-adjacent points $v, w \in V(G)$

$$d(f(v), f(w)) < d(v, w).$$

The behaviour of contractions is not very different from that of λ-contractions, especially for $\lambda = 1/2$.

Theorem 5. *Every contraction on a graph has a fixed point, or an inverted edge, or else an invariant cycle. The invariant points, edges and cycles span, all together, a complete subgraph.*

Proof. Like in the proof of Theorems 1 and 4, if the contraction f has no fixed points, we find an invariant thick cycle C. If U denotes the union of all fixed points, inverted edges and invariant thick cycles, and $u, v \in V(U)$, we have, like in the proof of the Corollary, $f^m(u) = u$ and $f^m(v) = v$ for some m.

Then

$$d(u, v) \geq d(f(u), f(v)) \geq \dots \geq d(f^m(u), f^m(v))$$

implies equality of all above terms, which in turn implies $d(u, v) = 1$.

Thus, the second part of the statement is true, and C must be a cycle spanning a complete sub-graph of G.

Note that every 1/2-contraction is a particular type of contraction.

Trees

We suppose now that G is a tree. In this case we can identify a very small invariant sub-graph, for any non-expanding mapping on G.

Theorem 6. *Every non-expanding mapping on a tree has a fixed point or an inverted edge.*

Proof. Assume that f is a non-expanding mapping on the tree G, without fixed points and without inverted edges. Then, by Theorem 2, there must exist an invariant proper thick cycle C, obtained as described in the proof of Theorem 2. Thus, C has vertices $a_0, a_1, \dots, a_{j-1}, a_j = a_0$ and $f(a_i) = a_{i+1}$ (mod j). Since G is a tree, each thick segment $S(a_i, a_{i+1})$ contains the single segment Σ_i. If $j = 1$, then C reduces to $\Sigma_0 = \Sigma_1$ and is not proper, so $j \geq 2$.

The segments $\Sigma_0 = v_0 v_1 \dots v_{k-1} v_k$ and $\Sigma_1 = w_0 w_1 \dots w_{k-1} w_k$ meet, because $v_k = a_1 = w_0 = f(v_0)$. Let v_q be the point of $\Sigma_0 \cap \Sigma_1$ closest to v_0. Since the distance from v_q to a_1 measured on Σ_0 and on Σ_1 must both equal $d(v_q, a_1) = k - q$, we have $v_q = w_{k-q}$.

If $q > k/2$, then $v_q, w_q = f(v_q), f^2(v_q), \dots, f^{j-1}(v_q), f^j(v_q) = v_q$ are vertices of a thick cycle which is a cycle, contradicting the definition of a tree.

If $q = k/2$, the previous thick cycle degenerates to the single point $v_q = w_q = f(v_q)$, and this contradicts our assumption.

Hence $q < k/2$. Since there is only one segment from v_q to a_1, we have $v_t = w_{k-t}$ for $q \leq t \leq k$. Thus, for even k, $v_{k/2} = w_{k/2}$, and $v_{k/2}$ is a fixed point, and for odd k, $v_m = w_{m+1}$ and $v_{m+1} = w_m$, where $m = \lfloor k/2 \rfloor$, and so $v_m v_{m+1}$ is an inverted edge.

Cacti

Let $\mathcal{P}$ be a family of pairwise disjoint paths (of positive lengths), and let $\mathcal{C}$ be a family of cycles such that any two of them have at most one point in common. Also assume that, if $P \in \mathcal{P}$ and $C \in \mathcal{C}$ meet, then their intersection is a single point, namely an endpoint of P. The graph

$$G = \bigcup \{H : H \in \mathcal{P} \cup \mathcal{C}\}$$

is called a *cactus* if it is connected and contains no cycle which does not belong to $\mathcal{C}$.

Even if a cactus resembles to a tree, we can not state that every cactus has a fixed point or an inverted edge. This becomes correct when adding invariant cycles.

Theorem 7. *Every non-expanding mapping on a cactus has a fixed point, or an inverted edge, or else an invariant cycle.*

Proof. We proceed like in the proof of Theorem 6, assuming the lack of fixed points and inverted edges and then finding a proper thick cycle, the segments $\Sigma_0 = v_0 v_1 \dots v_{k-1} v_k$ and $\Sigma_1 = w_0 w_1 \dots w_{k-1} w_k$, and the point $v_q = w_{k-q}$ in $\Sigma_0 \cap \Sigma_1$. Notice that this time $S(a_i, a_{i+1})$ may contain more than just one segment.

If $q > k/2$, then $v_q, w_q = f(v_q), f^2(v_q), \dots, f^{j-1}(v_q), f^j(v_q) = v_q$ are vertices of a thick cycle (with the segment $w_{k-q} w_{k-q+1} \dots w_{q-1} w_q$ between v_q and $f(v_q) = w_q$), which must be a cycle, otherwise contradicting the definition of a cactus. This cycle is invariant under f.

If $q = k/2$, the previous thick cycle degenerates to the single point $v_q = w_q = f(v_q)$, and this contradicts our assumption.

Suppose now $q < k/2$. Since f has no fixed point and no inverted edge, $v_{k/2} \neq w_{k/2}$ for even k, and $v_{\lfloor k/2 \rfloor} \neq w_{\lfloor k/2 \rfloor + 1}$ for odd k.

Let $s > k/2$ be the minimal integer such that $v_s = w_{k-s}$, and $r < k/2$ be the maximal integer such that $v_r = w_{k-r}$. Then

$$v_r v_{r+1} \ldots v_{s-1} v_s w_{k-s+1} \ldots w_{k-r-1} w_{k-r} \in \mathcal{C}.$$

Suppose $r < k - s$. (The case $r > k - s$ is analogous.) Put $z_i = f(w_i)$. The path $z_{k-r} z_{k-r+1} \ldots z_{k-s}$ joins w_r with w_s. This provides a cycle not in $\mathcal{C}$, or coincides with $w_r w_{r+1} \ldots w_s$ on its subpath $z_{k-s} z_{k-s+1} \ldots z_s$, i.e., $z_{k-s} z_{k-s+1} \ldots z_s = w_{k-s} w_{k-s+1} \ldots w_s$, which yields the fixed point $w_{\lfloor k/2 \rfloor}$ or the inverted edge $w_{\lfloor k/2 \rfloor} w_{\lfloor k/2 \rfloor + 1}$ (See Figure 3). This shows that $r = k - s$ and $z_{k-r} z_{k-r+1} \cdots z_{k-s} = v_s v_{s+1} \cdots v_r$, with $z_i = v_i$. Thus, $v_r v_{r+1} \ldots v_s w_{r+1} w_{r+2} \ldots w_{s-1} w_s$ is an invariant cycle under f.

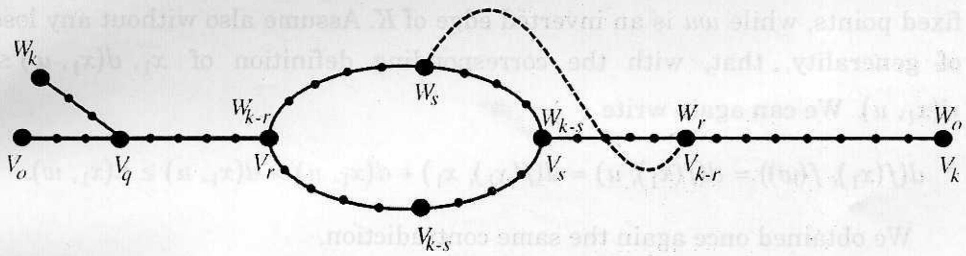

Figure 3

The number of possible invariant cycles is, however, small, as the next theorem shows.

Theorem 8. *Every non-expanding mapping on a cactus has at most one invariant cycle without fixed points. In case it has one such cycle, there neither exist fixed points, nor inverted edges.*

Proof. Suppose f is a non-expanding mapping on the cactus K, and C_1, C_2 are distinct cycles in K, both invariant under f and without fixed points. Let P be a shortest path joining the two cycles. It is understood that both $P \cap C_1$ and $P \cap C_2$ are single points, say x_1 and x_2. There is no path leaving C_1 at another point than x_1 and leading to C_2, as this would yield a cycle not in $\mathcal{C}$. The same is true if the indices are interchanged, so each path from C_1 to C_2 contains x_1 and x_2. Hence

$$d(v, w) = d(v, x_1) + d(x_1, x_2) + d(x_2, w)$$

for any two points $v \in C_1$, $w \in C_2$. But then $f(x_1) \neq x_1$ and

$$d(f(x_1), f(x_2)) = d(f(x_1), x_1) + d(x_1, x_2) + d(x_2, f(x_2)) > d(x_1, x_2),$$

contradicting the definition of a non-expanding mapping.

Assume now that C_1 is a cycle in K, invariant under f and without fixed points. Also, assume that w is a fixed point. A similar argument shows that, with the appropriate definition of x_1,

$$d(f(x_1), f(w)) = d(f(x_1), x_1) + d(x_1, w) > d(x_1, w),$$

contradicting again the definition of a non-expanding mapping.

Finally, assume that C_1 is a cycle in K, invariant under f and without fixed points, while wu is an inverted edge of K. Assume also without any loss of generality, that, with the corresponding definition of x_1, $d(x_1, w) \leq d(x_1, u)$. We can again write

$$d(f(x_1), f(w)) = d(f(x_1), u) = d(f(x_1), x_1) + d(x_1, u) > d(x_1, u) \geq d(x_1, w).$$

We obtained once again the same contradiction.

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References

[1] W. Imrich, S. Kristic and E. C. Turner, On the rank of fixed point sets of automorphisms of free groups, In: Cycles and Rays, NATO ASI Ser. C, Kluwer Academic Publishers, Dordrecht, 1990, pp. 113-122.

[2] W. Imrich, C. Turner, Endomorphisms of free groups and their fixed points, Math. Proc. Camb. Phil. Soc. 105 (1989), 421-422.

[3] N. Seifter, On automorphisms of infinite graphs with forbidden subgraphs, Combinatorica 4 (1984), 351-356.