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Pushing convex and other bodies through rings and holes

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Dedicated to Professor Dan Papuc on the occasion of his 80th anniversary

Abstract. Most convex bodies pass through a circle smaller than the section of their circumscribed cylinder. Most continua pass through arbitrarily small circles. In this paper we prove these and other similar results.

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Already in 1920 K. Zindler [7] observed that he can push a certain convex polytope, an affine image of the cube, through a circular hole of a planar wall, smaller than the section of a circumscribed cylinder (precise definitions will be given in the next section). Recently, we (with J. Itoh and Y. Tanoue) [1] did the same with the usual regular tetrahedron. What happens with the other convex bodies? Of course, not every convex body has this rather unexpected property. But we shall prove here that most of them enjoy it!
Evidently, these many convex bodies pass through a circle (a circular ring) smaller than the section of the circumscribed circular cylinder.
Thinking now about continua instead of convex bodies, we discover that most of them pass not only through sufficiently large, but also through arbitrarily

small circles. However, this is not true with respect to arbitrary circles! Nor with respect to arbitrarily small holes in the wall.

It is known [5] that most convex bodies can be carried with some circle (the body cannot escape from the circle, which surrounds it). This extends to continua in the stronger way that most of them can be carried with circles of any size.

Problems of a similar kind, regarding convex bodies passing through non-circular holes have also been investigated (see [1], [2], [4]).

Definitions and notation

Let $\mathcal{C}$ be the space of all continua in $\mathbb{R}^3$, and $\mathcal{K} \subset \mathcal{C}$ the subspace of all convex bodies, i.e., compact convex sets with interior points. Both spaces, endowed with the Pompeiu-Hausdorff distance h, are Baire spaces. As usual, we say that *most* elements of a Baire space enjoy property **P** if those elements not enjoying **P** form a first Baire category subset.

Let K be a convex body. An unbounded circular cylinder $C(K)$ is called *circumscribed cylinder* of K if $C(K)$ includes K and its orthogonal section has minimal radius. A convex body can have several circumscribed cylinders.

A *hole* H of the plane Π is a 2-dimensional convex body included in Π. We say that the continuum K *passes through the hole* H if there is a continuous function $f : [0,1] \to \mathcal{C}$ such that $f(0)$ and $f(1)$ are separated by Π, $f(t)$ is congruent to K and $f(t) \cap \Pi \subset H$ for all $t \in [0,1]$.

Let C be a circle in $\mathbb{R}^3$.

The convex body K is said to be *held* by by the circle C if, for some $K' \in \mathcal{K}$ congruent to K, there is no congruence $c_t : \mathbb{R}^3 \to \mathbb{R}^3$ depending continuously on t, such that c_0 is the identity, $c_1(K) = K'$ and $c_t(K) \cap C = \emptyset$ for all $t \in [0,1]$.

We say that $K \in \mathcal{C}$ *passes through the circle* C if for every $t \in C$, there exists a congruence $c_t : \mathbb{R}^3 \to \mathbb{R}^3$ depending continuously on t, such that $c_t(K) \cap C = \emptyset$ for all $t \in C$ and, for each point $x \in K$, the closed curves C and $\{c_t(x) : t \in C\}$ are linked.

We denote by $r_p(K)$ the radius of the smallest circle through which K can

pass.
With Zindler's observation in mind, V. Klee [3] raised in 1996 the problem of determining the smallest number r such that, whenever a convex body can pass through a circle of radius 1 it can also pass through a long cylinder of radius r, i.e.

$$r = \sup_{K\in\mathcal{K}} r_c(K)/r_p(K).$$

This is still open.

Convex bodies passing through circles

For an ellipsoid E, the smallest circle through which E can pass has the same size as the section of the circumscribed cylinder of E. The same is true for long right prisms. Are these examples exceptional?
Our first theorem confirms it.

Theorem 1. *All convex bodies, except those in a nowhere dense subset, pass through circles smaller than the section of their circumscribed cylinders.*

The aim of this section is to establish Theorem 1. This needs some preparation.
Let Ξ_K denote the cylinder with the z-axis as symmetry axis, congruent to a circumscribed cylinder of $K \in \mathcal{K}$, and let $r_c(K)$ be their radius.
Suppose now that the vertical cylinder Z, verifying in cylindric coordinates (ρ, α, z) the equation $\rho = r_c(P)$, is itself circumscribed to the polytope P. Let P_ξ be the intersection of P with the plane $z = \xi$.
For every ξ, the set $P_\xi \cap Z$ is finite. If $P_\xi \cap Z \neq \emptyset$, then every point of $P_\xi \cap Z$ (which must be a vertex of P_ξ if the convex polygon P_ξ is not degenerate) is either a vertex of P or an interior point of an edge of P lying entirely on Z.
Let $J = \{\xi : P_\xi \cap Z \neq \emptyset\}$. This set is a finite union of closed intervals, each of which is possibly reduced to a single point.
If $\xi \in \mathrm{bd}J$ then $P_\xi \cap Z \subset V(P)$, where $V(P)$ denotes the vertex set of P, but the converse is in general not true.
The following criterion was proved in [6].

Lemma. *If, for every* $\xi \in \text{int}J$, $(0,0,\xi) \notin \text{conv}(P_\xi \cap Z)$, *then* $r_p(P) < r_c(P)$.

Proof of Theorem 1. Let $\mathcal{U} \in \mathcal{C}$ be open, and choose a polytope $P \in \mathcal{U}$, with vertices in general position, in particular without parallel or orthogonal line-segments joining vertices.

Consider Ξ_P and assume it is circumscribed to P. Then

$$(0,0,\xi) \in \text{conv}(P_\xi \cap Z)$$

is possible for some $\xi \in \text{int}J$ only if $P_\xi \cap Z$ consists of two points $(r_c(P), \alpha, \xi)$ and $(r_c(P), \alpha + \pi, \xi)$ symmetric with respect to the z-axis, one of which, say the second, lies in the interior of a vertical edge of P.

Replace the polytope P by the polytope $Q \in \mathcal{U}$ defined as the convex hull of

$$(V(P) \setminus \{(r_c(P), \alpha, \xi)\}) \cup$$

$$\cup \{(r_c(P), \alpha - \varepsilon, \xi + \varepsilon), (r_c(P), \alpha + \varepsilon, \xi), (r_c(P), \alpha - 2\varepsilon, \xi - \varepsilon)\},$$

for adequately small $\varepsilon > 0$. Then $\Xi_P = \Xi_Q$, and the condition of the Lemma is satisfied for Q.

Thus, for a circle C of radius smaller than $r_c(Q)$, there is a congruence c_t such that $c_t(Q) \cap C = \emptyset$ for all t and, for each $x \in Q$, the curves C and $\{c_t(x) : t \in C\}$ are linked.

This remains true for all convex bodies at Pompeiu-Hausdorff distance at most η from Q, if $\eta > 0$ is chosen smaller than the positive number

$$\inf\{\|x - y\| : x, t \in C, y \in c_t(Q)\}.$$

Hence, the statement is proven.

□

Continua passing through holes

Theorem 2. *Every convex body can be approximated by continua passing through any hole.*

Proof. Let $K \in \mathcal{K}$ and $\eta > 0$.
Let $\Pi_1, ..., \Pi_n$ be the parallel planes

$$\Pi_i = \{x : \langle x, v\rangle = a + 2\eta i\}$$

which meet K, where v is an orthogonal unit vector. So, the distance between consecutive planes is 2η.

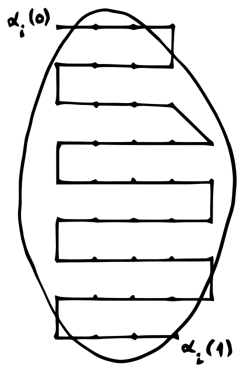

Figure 1

Every section $K_i = K \cap \Pi_i$ can be approximated by an arc $\{\alpha_i(t) : 0 \leq t \leq 1\}$ parametrized proportionally to arc-length, connecting grid points (of the planar grid with distance 2η between neighbouring points) as shown in Fig. 1. Let $a_i = \alpha_i(0)$ and $b_i = \alpha_i(1)$. We transform these arcs by moving each point $\alpha_i(t)$ to the new position $\beta_i(t) = \alpha_i(t) + \eta t v$, such that $\beta_i(0) = \alpha_i(0)$ and $\|\beta_i(1) - \alpha_i(1)\| = \eta$. Now join $\beta_i(1)$ with $\beta_{i+1}(0)$ by a line-segment σ_i. Put $\sigma_n = \emptyset$.
In this way we obtain an arc $\Gamma = \cup_{i=1}^{n}(\beta_i([0, 1]) \cup \sigma_i)$ with $h(K, \Gamma) < 4\eta$. The intersection of Γ with any plane parallel to Π_1 is a single point or empty. This $\Gamma \in \mathcal{C}$ is the set we are looking for.
It is obvious that, locally, a movement of Γ in direction of a line-segment in Γ containing $\beta_i(t)$ and lying in $\beta_i([0, 1])$ or σ_i for suitable i and t allows Γ to pass through a prescribed point of the hole. No rotation is ever necessary, just translations.

□

While for convex bodies, passing through a circular ring and passing through a circular hole in a wall are equivalent adventures, this is not so for arbitrary continua. The next result illustrates this, by showing that the "spectacular" result of Theorem 4 is not true if passing through a circle is replaced by passing through a circular hole.

Theorem 3. *The continua passing through arbitrarily small holes form a nowhere dense set.*

Proof. Let $M \in \mathcal{C}$ and $\varepsilon > 0$. Consider a cube of side-length ε and let A be the union of all 8 line-segments joining its centre with the vertices.
It can be checked that A cannot be approximated by any continuum passing through an arbitrarily small hole of some plane. (This becomes wrong if the cube is replaced by a regular octahedron!)
Let A' be a congruent copy of A such that $M \cap A'$ is a single point. Then the continuum $M' = M \cup A'$ is at Pompeiu-Hausdorff distance less than 2ε from M. Neither M', nor the continua in a whole neighbourhood of M', can pass through any hole in the wall which is small compared with ε.

□

Continua passing through circles

Theorem 4. *Most continua pass through arbitrarily small circles, i.e. for any $\varepsilon > 0$, they pass through a circle of diameter less than ε.*

Proof. Let $\mathcal{C}_\eta \subset \mathcal{C}$ be the set of all continua which pass through a circle of radius at most $\eta > 0$. We first prove that $\mathcal{C}_\eta$ is dense in $\mathcal{C}$.
Let $M \in \mathcal{C}$. Let $Z \subset 8\eta\mathbb{Z}^3$ be a set satisfying $h(Z, M) < 8\eta$. By joining various pairs of points of Z at distance 8η we get a geometric connected graph $G \in \mathcal{C}$ also satisfying $h(G, M) < 8\eta$. Now, we approximate G by a Jordan arc. Although this is not difficult, we give a proof.
First remove an edge of a cycle, and repeat this as long as cycles exist in G. We obtain a tree T spanning G, with $h(T, M) < 8\eta$.
Now, let T' be an embedding of T in a plane Π realized with line-segment as edges. There are many ways of performing this embedding e. We shall choose one which respects the following requirement. For any vertex v of T,

let the neighbourhood $N(e(v))$ of $e(v)$ consist of the vertices $v_1, ..., v_k$ $(k \leq 6)$ lying in this order around $e(v)$ in Π; it is required that, if $k \geq 4$, then

$$\langle e^{-1}(v_i) - v, e^{-1}(v_{i+1}) - v \rangle = 0,$$

i.e. the angle between the edges $(v, e^{-1}(v_i))$ and $(v, e^{-1}(v_{i+1}))$ should be $\pi/2$ $(i = 1, ..., k;\ v_{k+1} = v_1)$. There is no difficulty in arranging that e fulfils this requirement. Now let $\varepsilon > 0$ be very small, and consider the closed Jordan curve

$$J' = \{x \in \Pi : d(x, T') = \varepsilon\}.$$

When going along J' one "visits" each vertex v' of T' $d(v')$ times. The curve J' can be transferred to a closed Jordan curve J in $\mathbb{R}^3$ consisting of points at distance exactly η from T in a quite obvious way. Every vertex v of T is visited $d(v)$ times, and at each visit the distance to v is η or $\eta\sqrt{2}$. From the visit of a vertex v to the visit of a neighbour w of v, the curve J may need to spiral on the cylinder of radius η and axis vw. Thus, $h(J, T) = \eta$ and $h(J, M) \leq 9\eta$.

Now, the removal of a small piece from this curve produces a Jordan arc J'' and $h(J'', M) \leq 9\eta$. Obviously, this arc passes through a circle of radius η. So, $\mathcal{C}_\eta$ is dense in $\mathcal{C}$.

Of course, any continuum sufficiently close to J'' passes through the same circle. So, $\mathcal{C} \setminus \mathcal{C}_\eta$ is nowhere dense. This implies that most continua lie in

$$\mathcal{C} \setminus \cup_{n=1}^{\infty} (\mathcal{C} \setminus \mathcal{C}_{1/n}) = \cap_{n=1}^{\infty} \mathcal{C}_{1/n},$$

i.e. pass through arbitrarily small circles. □

Theorem 5. *Consider a circle. The (second category) set of all continua passing through that circle is not dense in* $\mathcal{C}$*.*

Proof. Let C and $\mathcal{C}^*$ be the given circle and set of continua. Of course, $\mathcal{C}^*$ is of second category, containing all continua of diameter smaller than diam $C/2$. However, no continuum in a whole neighbourhood of the arc A depicted in Fig. 2 will pass through the circle C of Fig. 2.

□

Continua held by a circle

All convex bodies, except those in a nowhere dense subset, can be held using a circle [5]. Here the holding circle strongly depends on the convex body to be held. Things get easier when continua should be held.

Theorem 6. *Most continua can be held by any circle.*

Proof. We prove that those continua not being held by any circle of diameter at least $\varepsilon > 0$ form a nowhere dense set $\mathcal{C}_\varepsilon$.

Let M be a continuum. Take $\eta > 0$. Now the arc A of Fig. 2 has diameter $\eta/2$ and its knot is small compared with ε. No circle C of radius at least ε positioned as in Fig. 2 can be removed without deformation. Glue a congruent copy A' of A to M by identifying an exposed point of conv$M$ with the endpoint $a \in A$ such that $M \setminus \{a\}$ and $A' \setminus \{a\}$ be separated by a plane through a, and obtain a continuum M'. We have $h(M, M') < \eta$. Every continuum in a small neighbourhood of M' is held by C. Thus, $\mathcal{C}_\varepsilon$ is nowhere dense.

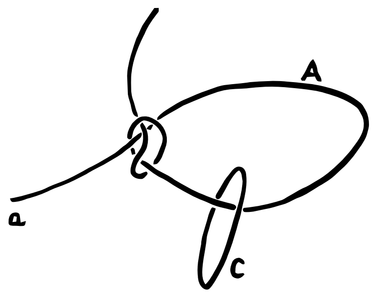

Figure 2

Hence $\cup_n \mathcal{C}_{1/n}$ is of first Baire category and contains precisely all continua not being held by any circle.

□

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