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Paolo Pizzetti: The forgotten originator of triangle comparison geometry

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Abstract

The theorem comparing the angles of two geodesic triangles with the same side lengths lying on surfaces with different curvatures, commonly attributed for the two-dimensional case to A.D. Alexandrov (1948) and for the n-dimensional Riemannian case to V.A. Toponogov (1959), goes back, for the two-dimensional case, to Paolo Pizzetti (1907b). Besides suggesting a correction of the historical narrative regarding the development of comparison geometry, the present note also mentions possible reasons for the oversight.

Riassunto

Il teorema di confronto fra gli angoli di due triangoli geodetici con uguali lati descritti sopra superficie di curvature diverse, comunemente attribuito per il caso bidimensionale a A.D. Alexandrov (1948) e per il caso n-dimensionale riemanniano a V.A. Toponogov (1959), risale, per il caso bidimensionale, a Paolo Pizzetti (1907b). Oltre a proporre una correzione storica riguardante l'origine e i primi sviluppi della geometria di confronto, la presente nota indica anche alcuni possibili motivi per la predetta dimenticanza.

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1. Introduction

There are very few accounts of the origins and development of comparison geometry. Except for remarks hidden in the editors' preface of Grove and Petersen [1997], the

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proceedings of the Special Year in Differential Geometry held in Berkeley, CA, 1993–1994, the names associated with the main triangle comparison theorem in Burago et al. [2001], the seven lines of the history of the triangle comparison theorem in Chavel [2006], the obituaries of A.D. Alexandrov [Borisov et al., 1999] and V.A. Toponogov [Aleksandrov et al., 2006], the main sources for a history of differential geometry (mostly in the second half of the 20th century) are [Berger, 1998; Petersen, 1999]. There are only two pages relevant to the history of triangle comparison geometry in the former, 52 and 68, and one in the latter, 325.

The triangle comparison theorems (going by the names of the Aleksandrov Comparison Theorem for the Angles of a Triangle [Toponogov, 2006, Theorem 3.9.1, p. 189] in the two-dimensional case, and the Alexandrov–Toponogov theorem [Chavel, 2006, p. 400] or the Cartan–Alexandrov–Toponogov theorem [Burago et al., 2001, Theorem 6.5.6, p. 240] in the higher-dimensional case) have two forms: (α) one comparing lengths, and stating[1] that given two geodesic triangles T_1 and T_2 with two sides of the same length and the angle between those two sides of the same measure, lying on two surfaces S_1 and S_2, the Gaussian curvature k_1 of the former being everywhere greater than or equal to the maximum of the Gaussian curvature k_2 of the latter, the third side of triangle T_1 will be less than or equal to the third side of triangle T_2, and (β) one comparing angles, and stating that, under the same conditions on S_1 and S_2, if the triangles T_1 and T_2 have congruent sides, then the angles of T_1 are greater than or equal to their counterparts in T_2. Their significance is perhaps best expressed in Petersen [2006, p. 333]: “Toponogov’s theorem is a very useful refinement of Gauss’s early realization that curvature and angle excess of triangles are related.”

If one were to piece together an account of the history of these theorems from the literature, one would have to conclude that the names to be attached to this particular theorem of comparison geometry are E. Cartan, A.D. Alexandrov, and V.A. Toponogov—this being why M. Gromov in 1987 called certain spaces whose curvature is defined in comparison to spaces of curvature k via the triangle comparison theorem by the initials of these three geometers: CAT(k) spaces. Cartan in [Cartan, 1946, Théorème in §230] proved the theorem only for infinitesimal triangles, and claimed that it goes back to [Darboux, 1894, Livre VI, Chapitre VI]. Looking for Darboux’s contribution, one finds that his aim in that chapter is to prove what he calls Gauss’s theorem on the defect of a triangle on a surface, proved in Darboux [1894, p. 127ff, Section 611–612], which is today referred to as a form of the Gauss–Bonnet theorem, to be found in do Carmo [1976, p. 279] or Toponogov [2006, Corollary 3.7.1]. Alexandrov [1948] proves it in a framework significantly more general than that of Riemannian geometry, for the two-dimensional case, whereas Toponogov [1959] lifts the dimension restriction (and thus the Gaussian curvature becomes sectional curvature) but stays within the Riemannian framework. The proofs of Pizzetti and of Alexandrov and Toponogov cannot be meaningfully compared, given that, although the theorems are phrased similarly, the definitions of the terms involved are so different, that the proofs could not have displayed any similarity. H.E. Rauch’s comparison theorem [Rauch, 1951] also figures prominently in these narratives, as “by triangle comparison theorems we mean global forms of the Rauch comparison theorem” ([Chavel, 2006, p. 399]; a similar account is presented in Cheeger and Ebin [1975]), and “although a name has been lacking for this beautiful and most geometric branch of riemannian geometry, its history can be traced back to the nineteenth century. It did not take root however, until the 1930’s,

[1] The statements will be given, for reasons of simplicity, for the two-dimensional case and without the necessary technical conditions for the positive curvature case.

through the work of H. Hopf, Morse and Schoenberg, Myers and Synge. The real breakthrough came in the 1950's with the pioneering work of Rauch and the foundational work of Alexandrov, Toponogov and Bishop" [Grove and Petersen, 1997, p. ix].

2. Paolo Pizzetti and the triangle comparison theorems

Paolo Pizzetti was born on July 24, 1860 in Parma and died on April 14, 1918 in Pisa. He studied engineering in Rome, graduating in 1880, stayed on as an Assistant to work with Giuseppe Pisati and Enrico Pucci on their remarkable absolute determination of gravity. Leaving the experimental period of his life behind, he became in 1886 Associate Professor of Geodesy (*professore straordinario di Geodesia*) at the University of Genoa, where he stayed until 1890, when he became Professor of Mechanical Geodesy at the University of Pisa, a position he held until his death. From his multifaceted output, we mention five major areas of concentration: (i) the theory of errors, for which [Pizzetti, 1892] is a representative piece, (ii) geodetic and astronomical refraction, on which he wrote, beside a multitude of papers, a book [Pizzetti, 1905], and the chapter on geodesy [Pizzetti, 1907f] in the famous *Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen*, (iii) the mechanical theory of the shape of planets, the results of which are collected in Pizzetti [1913], (iv) Pizzetti's formula relating the spherical average of an analytic function about a point to the values of its iterated Laplacians at that point, which generated a whole literature of "Pizzeti-like formulas" in potential theory, and was noticed by three differential geometers as well [Willmore, 1981; Gray and Willmore, 1982; Kowalski, 1982], who extended it to Riemannian manifolds, the only instances in which differential geometers do mention his name after a sporadic mention in Olevsky [1944], and (v) Pizzetti's triangle comparison theorems. There were two obituaries, by his geodesy colleague Vincenzo Reina [Reina, 1918], and, using Reina's obituary, by Bryan [1918]. A crater on the far side of the moon is named after him.

Pizzetti's contributions to triangle comparison geometry were initially published during one year, 1907, in a five-paper sequence [Pizzetti, 1907a–e]. Ten years later he came back to the subject in Pizzetti [1917], as an addendum to a paper by Severi [1917], in which the latter addresses, without mentioning Pizzetti's work, several questions, initiated by T. Levi-Civita, regarding the relationship between curvature and angles of infinitesimal triangles and quadrilaterals. The angle comparison version (β) for non-negative curvatures is proved in Pizzetti [1907b] and Pizzetti [1907e, pp. 260, 262]; the length comparison version (α) is proved in Pizzetti [1907d, pp. 454–455], and Pizzetti proves in Pizzetti [1907d, p. 456] a third version as well, corresponding—the way (α) and (β) correspond to the side–side–side and to the side–angle–side congruence cases for triangles—to the angle–side–angle congruence case, by stating that, under the conditions of the comparison theorems, if T_1 and T_2 have a congruent side, as well as correspondingly congruent adjacent angles, then the third angle of T_1 will be greater than or equal to that of T_2. And Pizzetti does not stop here. His main concern being theoretical geodesy, the original motivation being to find the upper limit of the errors being committed in the approximate numerical resolution of geodesic triangles (which are being replaced by their Euclidean counterparts), he provides [Pizzetti, 1907e, p. 262; Pizzetti, 1907a, pp. 282–286; Pizzetti, 1907c, 1917] in (β) upper estimates for the difference between the two corresponding angles, depending on the lengths of the sides of the two triangles and the maximal value of one of the curvatures. Such estimates exist in the post-Pizzetti literature only in the case of the (α) form comparing lengths, in Alexandrov [1950, Th. 2, p. 189 (English ed.)], so Pizzetti's estimate can be said to have been not only removed from the memory of, but lost from the realm of results actively known to present-day differential geometry. One can ask whether Pizzetti learned some differential geometry from Luigi Bianchi, his colleague in Pisa. There is no mention of Bianchi anywhere in his papers or obituary, and the differential geometry used is of the classical variety, which does not require any look into Bianchi's papers. In Pizzetti [1907a], he cites Christoffel, Gauss, Bonnet, and Darboux [Darboux, 1894, Section 629], and his methods are entirely within the spirit of Darboux's book.

Another area of intersection of Pizzetti's and Alexandrov's interests is in the geometry of polyhedra, on which Pizzetti wrote [Pizzetti, 1898], sketching the notions of *spherical image* and of *flexible polyhedra* found in Alexandrov [1950, English edition: 1.5, 5.2].

3. Outline of Pizzetti's proof

We present here a brief outline of Pizzetti's proof of (β) from Pizzetti [1907e]. If we choose for the surface S a system of polar coordinates with pole P, with u designating the polar geodesic radius and v the polar angle (see [do Carmo, 1976, pp. 286–287]), then the first fundamental form of S can be written as $ds^2 = du^2 + g^2 dv^2$ (see [do Carmo, 1976, p. 287, Proposition 3]). The Gaussian curvature (which Pizzetti calls *curvatura assoluta*) k, and the geodetic curvature in M (a point of geodesic polar coordinates (u, v)) of the geodesic circle (see [do Carmo, 1976, p. 287]) with center in P and passing through M, geodesic curvature to be denoted by $-\frac{1}{R}$, are then given by

$$k = -\frac{1}{g}\frac{\partial^2 g}{\partial u^2} \quad \text{and} \quad \frac{1}{R} = \frac{1}{g}\frac{\partial g}{\partial u}. \tag{1}$$

In the geodesic triangle PQM, with sides QM of length s, PQ of length c, and PM of length u, with θ the measure of $\widehat{PMQ}$, and with $-\frac{1}{R_1}$ the geodesic curvature in M of the geodesic circle of center Q and passing through M, Pizzetti shows that, by keeping P and Q fixed, one gets the differential equations for θ, understood as a function of u and s,

$$\frac{\partial \cos\theta}{\partial s} = \frac{1}{R} - \frac{\cos\theta}{R_1} \quad \text{and} \quad \frac{\partial \cos\theta}{\partial u} = \frac{1}{R_1} - \frac{\cos\theta}{R}, \tag{2}$$

and the differential equation for the geodesic curvature of the geodesic circle of radius u

$$\frac{\partial R}{\partial u} = 1 + kR^2, \qquad \text{with } R(0) = 0. \tag{3}$$

Assume now that the Gaussian curvature of S satisfies

$$k_1 > k > k_2, \tag{4}$$

where k_1 and k_2 are real constants (in case $k_1 > 0$, Pizzetti notes that one needs to impose the condition $\leqslant \pi/(2\sqrt{k_1})$ on the values of u). Let (3)$'$ denote the differential equation (3) in which R has been replaced by r_1 and k by k_1. Setting $y = r_1 - R$, and computing the first three partial derivatives of y along u in $u = 0$, he determines that y must be positive and increasing on a certain interval $(0, u_1)$, for some $u_1 > 0$, and, noticing that, by (2),

$$\frac{\partial y}{\partial u} = ky(r_1 + R) + (k_1 - k)r_1^2, \tag{5}$$

he concludes that $\frac{\partial y}{\partial u}$ must be strictly positive whenever $y = 0$, so that y must be positive throughout (for if it ever took the value 0, it would have to be decreasing the first time it did so, so its derivative as a function of u could not have been positive). Since $y > 0$ means

$r_1 > R$, and, by the same reasoning, one gets $R > r_2$, where r_2 is given by the differential equation (3), in which R has been replaced by r_2 and k by k_2, he concludes that

Lemma 1. *If a surface S has its Gaussian curvature bounded from below by k_2 and from above by k_1, then the geodesic curvature of the geodesic circle of radius u on S is bounded from below by the geodesic curvatures of the geodesic circles of radius u drawn on a surface of constant Gaussian curvature k_2 and from above by the geodesic curvature of the geodesic circle of radius r on a surface of constant Gaussian curvature k_1.*

From here on, for the sake of simplicity, Pizzetti restricts his considerations to the case in which k_1 and k_2 are positive.

In the geodesic triangle PQM, keeping P and Q fixed, and considering θ once more as a function of u and s, with the substitutions $u + s = \alpha$ and $u - s = \beta$, he turns (2) into

$$\frac{\partial}{\partial\alpha}\left(\ln\sin\frac{\theta}{2}\right) = -\frac{1}{4}\left(\frac{1}{R_1} + \frac{1}{R}\right). \tag{6}$$

By noticing that the geodesic hyperbola given by $\beta =$ constant (the locus of all points on S for which the difference of the geodesic distances from P and Q is constant) intersects the geodesic arc PQ at a point A between P and A, such that $PA - AQ = \beta$, and that, when M coincides with A, angle θ becomes 180°, and thus $\ln\sin\frac{\theta}{2} = 0$, he gets from (6) (bearing in mind that the value of α in A is $PA + AQ = c$)

$$\ln\sin\frac{\theta}{2} = -\frac{1}{4}\int_c^{u+s}\left(\frac{1}{R} + \frac{1}{R_1}\right)d\alpha. \tag{7}$$

If one now repeats the same reasoning on the surface S' on which the minimum of the Gaussian curvature is greater than the maximum of the Gaussian curvature on S, and considers a geodesic triangle of sides c, u, s, the angle θ' opposite c in this triangle will satisfy (7) with θ' instead of θ and R' and R'_1 instead of R and R_1. By our Lemma, we know that $R' > R$ and $R'_1 > R_1$, and so $\ln\sin\frac{\theta'}{2} > \ln\sin\frac{\theta}{2}$, thus $\theta' > \theta$, proving (β).

Looking at a modern proof that is also in the context of 2-dimensional surfaces, such as the proof in Toponogov [2006, 3.9], one is struck by the fact that there are no similarities to Pizzetti's proof whatsoever. Toponogov's approach is one of synthetic geometry; no differential equation appears in it. Moreover, Pizzetti's results and proofs are local, whereas Toponogov's are global.

4. Possible reasons for oblivion

Why was all this forgotten? How could six different papers be ignored by everyone, until Pizzetti [1907e] was first mentioned as a precursor of Toponogov's theorem in Zamfirescu [2007, 2008], on its centennial anniversary? If one thinks that a possible reason is that the venues in which [Pizzetti [1907a–d, 1917] were published were not very prominent (although Toponogov does cite Fermi [1922], a paper that appeared 15 years later in the same journal in which Pizzetti [1907b–d, 1917] appeared), then there is a need to explain how Pizzetti [1907e], reviewed in *Jahrbuch für die Fortschritte der Mathematik*, which reproved by different means the results published in Pizzetti [1907a,b,c], published in one of the most prominent journals of the time—with papers by resonant names in the history of mathematics, such as J.W. Young, A. Vitali, H. Poincaré, N. Nielsen, J. Luröth, T. Levi-Civita, E. Borel, G. Fubini, E. Landau, in the same volume in which Pizzetti [1907e] appeared—got ignored.

The answer seems to us to be provided by the quotation above from Grove and Petersen [1997, p. ix]: differential geometry was not ripe for comparison geometry in 1907 or 1917. Else, Bianchi would have shown some interest in the work of not only his colleague in Pisa and fellow member of both the *Reale Istituto Veneto* and the *Reale Accademia dei Lincei*, but also his only-four-years-younger fellow *parmigiano*, and would have mentioned Pizzetti's results in the more than 1600 pages of his [Bianchi, 1923]. However, results by themselves do not deserve to be included in a textbook unless they fit nicely into some narrative. In Bianchi's time, the only narrative that had been suggested for triangle comparison geometry was in the context of approximations of triangulated maps in geodesy. That context, once of interest to Gauß (see [Scholz, 1992]), was no longer seen as belonging to differential geometry, which had long since emancipated itself from its rather humble origins. Triangle comparison geometry re-entered the horizon of interest when, in the wake of Alexandrov [1948], it was seen as a central instrument for providing a purely metric definition for notions resembling curvature in the absence of a C^2-structure (in fact, in the absence of any kind of differentiability).

Comparison geometry would have to wait for the 1930s, after Bianchi's death in 1928, to start making its first steps, and after World War II, when Alexandrov [1948] and Toponogov [1959] appeared, geometers would look for predecessors within the lineage of differential geometers. E. Cartan's and Alexandrov's references are all to the great classics of differential geometry. There was no room for an outsider, a discipline-crossing Italian self-taught differential geometer. We hope that this note will be noticed by the author of the yet-to-be-written history of differential geometry.

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