Hamiltonian properties of generalized pyramids

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Dedicated to Zdzisław Skupień on the occasion of his 70th birthday.

Abstract. We investigate here the hamiltonicity and traceability of a class of polytopes generalizing pyramids, prisms, and polytopes with Halin 1-skeleta.

Key Words. Pyramid, prism, Halin graph, hamiltonian.

1 Introduction

In this paper we introduce a class of polytopes including pyramids and prisms, generalizing those polytopes having Halin graphs as 1-skeleta. We call them k-pyramids, and investigate here their hamiltonian properties.

A polytope in $\mathbb{R}^3$ is said to be *hamiltonian*, or *traceable*, if its 1-skeleton has a hamiltonian cycle, respectively path. Two facets of a polytope will be called *neighbouring*, if they share a common edge. A polytope or one of its facets is called *simple*, if each of its vertices lies on precisely three edges of the polytope.

A polytope P is called a *k-pyramid*, if it has at most k pairwise disjoint simple facets $F_1, ..., F_k$, called *bases*, such that every other facet has some neighbouring base. We call the k-pyramid P *belted* if some pair of bases (if there are at least two) has no common neighbouring facet.

Of course, a pyramid is a particular case of a 1-pyramid, while any (combinatorial) prism is a particular case of a 2-pyramid.

The 1-skeletons of 1-pyramids are precisely the well-known Halin graphs. It is proven in [2] that they are all hamiltonian (even 1-hamiltonian, i.e. they are hamiltonian and remain hamiltonian when removing an arbitrary vertex). Is hamiltonicity preserved in k-pyramids, for larger values of k? This is the question we want to address here.

Other generalizations of Halin graphs and investigations of their hamiltonian properties have already been made by Skowrońska [6], Skowrońska and Sysło [7], Skupień [8], and Malik, Qureshi and Zamfirescu [5].

Consider a k-pyramid. Its 1-skeleton is a planar, 3-connected graph with at most k pairwise disjoint cycles called *basic cycles* which bound the bases. It is important to observe that this graph need not be cubic, but all vertices on the basic cycles have degree 3.

We say that a belted 2-pyramid P with basic cycles C_1, C_2 is *simply belted* if every vertex of the unique cycle disjoint from $C_1 \cup C_2$ in the 1-skeleton of P is of degree 3.

Let H be a Halin graph, v a vertex on its outer cycle (the basic cycle) and v_1, v_2, v_3 the neighbours of v. Then $H - v$ is called a *reduced Halin graph* and v_1, v_2, v_3 its *endpoints*. The following lemma will be used.

Lemma 1 [3]**.** *For any pair x, y of endpoints of a reduced Halin graph, there exists a hamiltonian path between x and y.*

This allows us to contract reduced Halin graphs appearing as subgraphs of a graph G to single vertices, without altering the hamiltonicity of G.

We shall use the following result.

Grinberg's Criterion [4]**.** *Let H be a hamiltonian cycle in a planar graph on n vertices and let f_j (g_j) be the number of j-gons inside (respectively outside) of H. Then we have*

$$\sum_{i=3}^{n} (i-2) \cdot (f_i - g_i) = 0.$$

Finally, the following simple fact will be of use.

Lemma 2. *Let G be a bipartite graph, its vertices coloured black and white. For G to be hamiltonian, the number of black vertices must be equal to the number of white vertices.*

2 Hamiltonicity of 2-pyramids

Our first main result is the following.

Theorem 1. *Every non-belted 2-pyramid is hamiltonian.*

Proof. Let G be the 1-skeleton of a non-belted 2-pyramid. If G is Halin, the hamiltonicity is known [2]. If not, we embed G in the plane such that the outer cycle is one of the basic cycles. Let us denote the inner basic cycle by A and the outer cycle by B. Let T be one of the trees in the forest F with leaves on $A \cup B$. Furthermore, let R_A (respectively R_B) be the union of all regions bounded by edges in $T \cup A$ (respectively $T \cup B$). Let $a_1, ..., a_m$ be the consecutive vertices of $T \cap A$, and $b_1, ..., b_n$ those of $T \cap B$. Since G is planar and 3-connected, no other tree of F has endpoints between a_i and a_{i+1} ($i = 1, ..., m-1$), or between b_j and b_{j+1} ($j = 1, ..., n-1$).

Thus, the boundary of the closure of $R_A \cup R_B$ is the union of the paths $a_1 a_2 ... a_m$, $b_1 b_2 ... b_n$, a path P_1 joining a_1 and b_1, and another path P_2 joining a_m and b_n.

The intersection of the closures of R_A and R_B is a path $P_3 = c_1 c_2 ... c_k$ joining a vertex c_1 on P_1 and a vertex c_k on P_2.

Let $d \in T - P_3$ be a neighbour of $c_i \in P_3$. The union of all paths starting at d, not containing c_i and ending as soon as $A \cup B$ is reached is a subtree of T. By

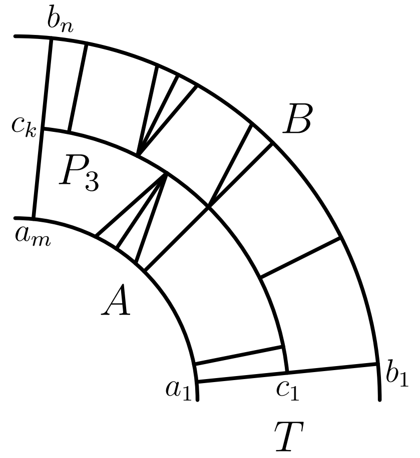


Fig. 1

Lemma 1, this subtree plus the path it spans in $A \cup B$ can be contracted to a single vertex (on $A \cup B$). After performing all such contractions, T looks like in Fig. 1.

Now, each triangle can be contracted to an edge between P_3 and $A \cup B$ because if the graph obtained after contraction is hamiltonian, then so was it before the contraction. Thus T becomes even simpler, as depicted in Fig. 2. The paths $a_1a_2...a_m$, $b_1b_2...b_n$ become $a'_1a'_2...a'_r$, $b'_1b'_2...b'_s$, respectively.

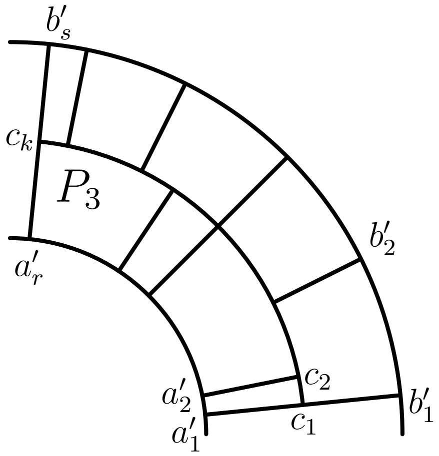


Fig. 2

For this tree T, obtained after contractions, the subgraph $T \cup a'_1...a'_r \cup b'_1...b'_s$ spanned by T admits a hamiltonian path from a'_1 to b'_s, namely $a'_1...a'_rc_k...c_2c_1b'_1...b'_s$. It admits a second one from b'_1 to a'_r, namely $b'_1...b'_sc_k...c_2c_1a'_1...a'_r$. And it admits a third hamiltonian path from a'_1 to a'_r if c_2 is adjacent to a vertex in A, and from b'_1 to b'_s if c_2 is adjacent to a vertex in B. In the situation of Fig. 2, c_2 is adjacent to a vertex in A and T has the hamiltonian path $a'_1c_1b'_1...b'_sc_kc_{k-1}...c_2a'_2...a'_r$.

If F contains an even number of trees, then paths of the first two types in each subgraph spanned by a tree can be combined to result in a hamiltonian cycle of G.

If F has oddly many trees, then one of them has a hamiltonian path of the third type, joining vertices on the same cycle, either A or B (see Fig. 3). Then the other hamiltonian paths in the remaining tree-spanned subgraphs can be alternately chosen of the first two types, to complete a hamiltonian cycle in G. $\square$

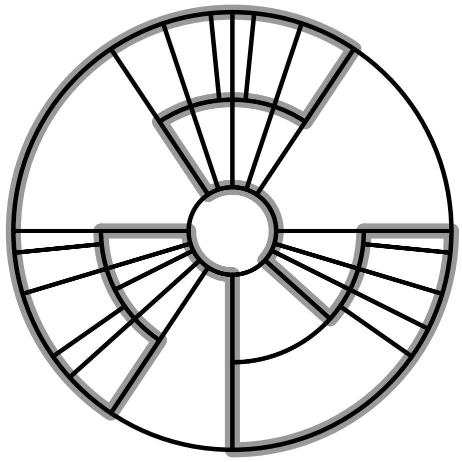


Fig. 3

Theorem 2. *Every simply belted* 2*-pyramid is hamiltonian.*

Proof. After the reductions mentioned in the preceding proof we arrive at a graph as described in Fig. 4. We obviously may also perform (possibly multiple times) the reduction of Fig. 5, which preserves non-hamiltonicity, so the graph becomes as shown in Fig. 6.

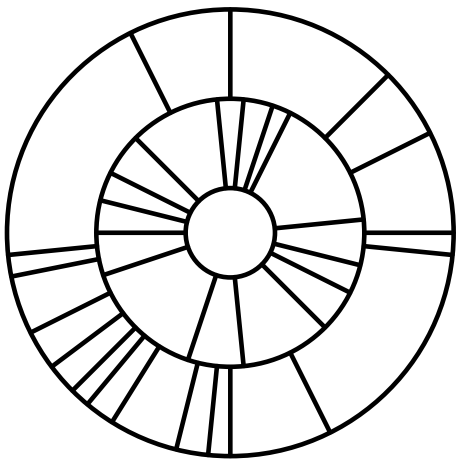


Fig. 4

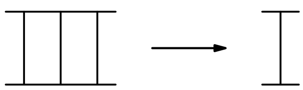


Fig. 5

This graph G has its two basic cycles C_1, C_2 and a third disjoint cycle (the belt) C_3. The faces of G different from the bases are quadrilaterals, pentagons, or hexagons.

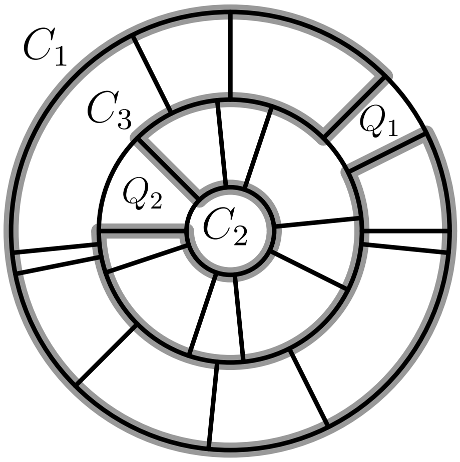


Fig. 6

The existence of a quadrilateral Q_i between C_i and C_3 (for each $i \in \{1, 2\}$) yields the existence of a hamiltonian cycle H_1 in the subgraph spanned by $C_1 \cup C_3$, using $C_1 \cup C_3$ minus two edges of Q_1, which in turn yields the existence of a hamiltonian cycle in G using $H_1 \cup C_2$ minus two edges of Q_2 (see Fig. 6).

The existence of two quadrilaterals between C_1 and C_3 (or between C_2 and C_3) also yields the existence of a hamiltonian cycle in G, see Fig. 7.

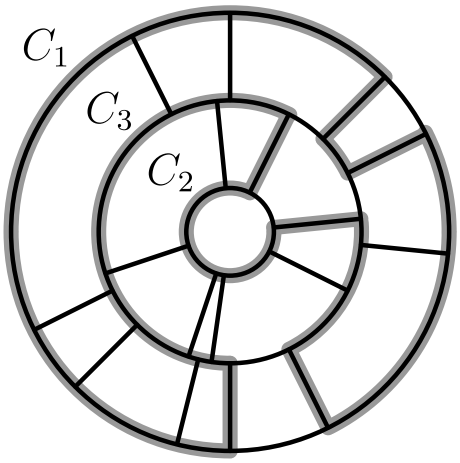


Fig. 7

If none of the two mentioned situations occurs, then all faces distinct from the bases, with at most two exceptions (a quadrilateral and a hexagon with a common edge on the belt), must be pentagons. Then a hamiltonian cycle is depicted in Fig. 8. □

Corollary. *Every simple* 2*-pyramid is hamiltonian.*

The following observation will be of use.

Remark. Consider (i) contracting a reduced Halin graph, (ii) contracting triangles into edges as shown in Figs. 1 and 2, and (iii) contracting edges as in Fig. 5.

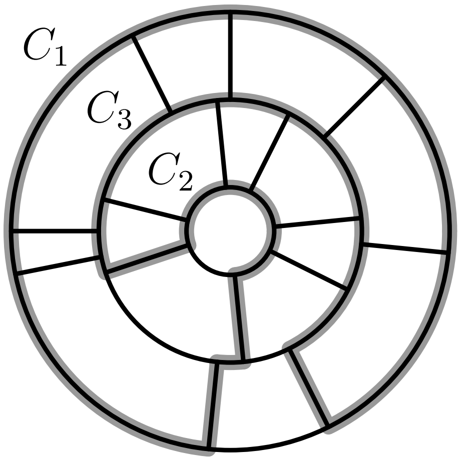


Fig. 8

If any of these contractions leads to a traceable (hamiltonian) graph, then so was the graph before the contraction.

We conclude the description of 2-pyramids by treating the general case.

Theorem 3. *Every* 2*-pyramid is traceable, but not necessarily hamiltonian.*

Proof. We show that not all 2-pyramids are hamiltonian. In Fig. 9 we present a belted 2-pyramid, which is non-hamiltonian. Indeed, let this graph be denoted by G and assume it has a hamiltonian cycle H. Two situations may occur: either H uses just two opposite edges of the basic cycle C_2 (see Fig. 9), or it uses all edges of C_2 but one. If H uses the edges ab and cd (avoiding bc and ad), we delete bc and da, contract both ab and cd to single vertices, and colour these black. Taking the colouring of G in Fig. 9 into account, we now have a bipartite graph which is non-hamiltonian by Lemma 2, as there are two more black vertices than white ones. If H uses the edges ab, bc and cd, we delete da and contract the path $abcd$ to a single vertex, which we colour black. Again, by using the colouring of G in Fig. 9, we now have a bipartite graph which is non-hamiltonian by Lemma 2, as there is one more black vertex than white ones.

We prove now the traceability of any belted 2-pyramid P. We use the full power of Lemma 1 and contract as in the proof of Theorem 1 reduced Halin graphs in the 1-skeleton of P, transform triangles into edges, then apply the reductions of Fig. 5, to finally obtain a graph G spanned by the basic cycles C_1, C_2 and the belt B. We apply the Remark to G.

Let $u, v \in C_1$ be adjacent. Let $u', v' \in B$ be neighbours of u, v, respectively.

If the facet containing the vertices u, v, v', u' has any further vertex, then let w be that one adjacent to u'. If not, then let w be the vertex of B adjacent to v' and distinct from u'. Further, let w' be the neighbour of w on C_2.

A hamiltonian path of G is now composed by a path spanning C_1 from u to v, which continues via v', w, w' and visits the remaining vertices of C_2, and another path starting in u, containing u' and visiting all vertices of B which are not on the first path. □

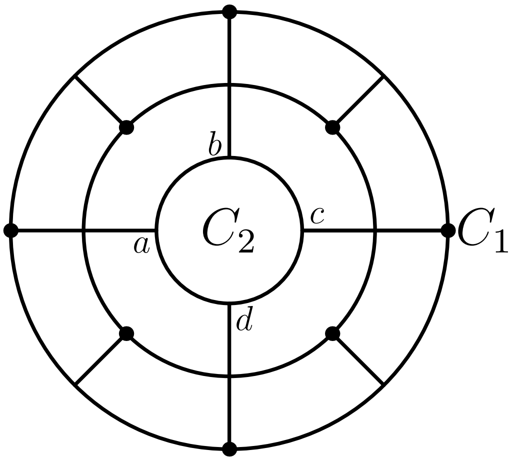


Fig. 9: A belted, non-hamiltonian 2-pyramid.

3 Traceability of 3-pyramids

We continue here our investigation by considering 3-pyramids.

Theorem 4. *Every non-belted* 3*-pyramid is traceable.*

Proof. If the (in this section always non-belted) 3-pyramid has at most two basic cycles, we apply Theorem 1. If it has three, C, C_1, C_2, let one of them, say C, be the outer cycle.

As in the proof of Theorem 1, using Lemma 1, we transform trees that have their leaves on precisely two basic cycles into trees of the form shown in Fig. 2. In such a tree we described in the proof of Theorem 1 three types of hamiltonian paths. We now use only the first two of them, and replace the whole tree by a single edge between the two basic cycles, keeping in mind that they can be visited by the hamiltonian path only in the ways shown in Fig. 10.


Fig. 10

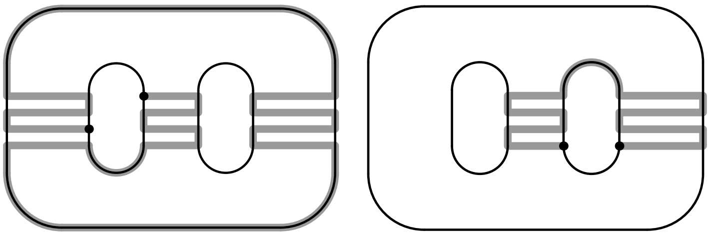
Thus, the forest of all trees between any two of the cycles C, C_1, C_2 becomes a set of equally many consecutive edges between those cycles. We distinguish several cases according to the parity of the number of edges in each set, i.e. trees in each forest.

Situation jkl means that j edges have their endpoints on $C \cup C_1$, k edges have their endpoints on $C_1 \cup C_2$ and l have them on $C \cup C_2$. The numbers j, k, l will be taken modulo 2, so $j, k, l \in \{0, 1\}$. There might also exist one or two trees having their leaves on all three basic cycles.

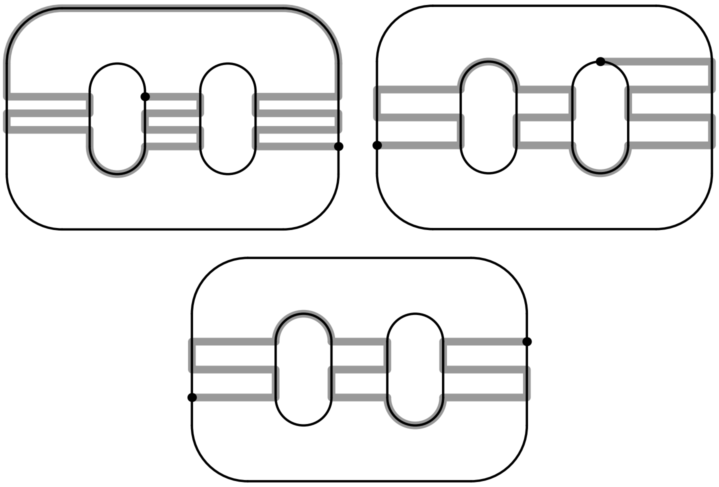
Case I: there are no trees with vertices on all special cycles.

In this case, there are four essentially different situations jkl, namely 000, 100, 110 and 111.

Situation 000: if $k \geq 2$, we find a hamiltonian path as shown in Fig. 11. If $k = 0$, the graph is hamiltonian (see Fig. 12). We conclude Case I with Situations 100, 110 and 111: see Fig. 13, 14 and 15, respectively.



Figs. 11 and 12: Situation 000 for $k \geq 2$ and $k = 0$.



Figs. 13–15: Situations 100, 110 and 111.

Case II: there exists exactly one tree with vertices on all special cycles.

Such a tree looks after contracting reduced Halin graphs as depicted in Fig. 16. There can be degeneracies of this aspect, but they can be deduced from the general one.

For Situation 000, see Fig. 17.

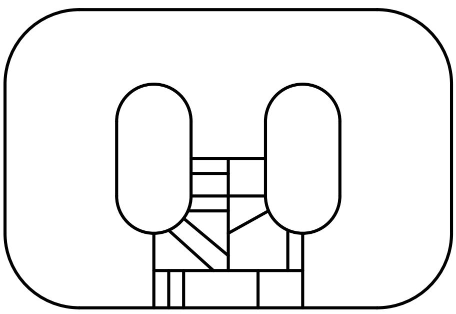


Fig. 16

For situations $10x$ and $11x$, see Fig. 18 and 19, respectively.

Case III: there exist exactly two trees with vertices on all special cycles.

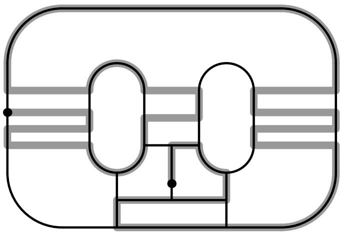
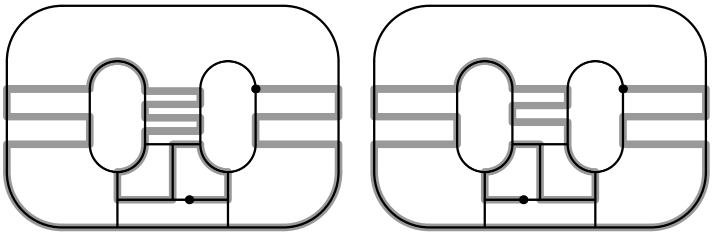
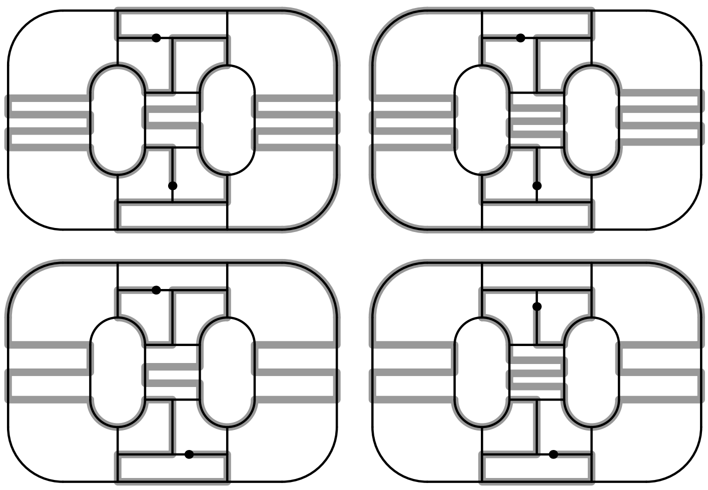


Fig. 17



Figs. 18 and 19: Situations $10x$ and $11x$.

For situations 000, 100=010=001, 110=101=011 and 111: see Fig. 20, 21, 22 and 23, respectively. □



Figs. 20–23: Situations 000, 100=010=001, 110=101=011 and 111.

Theorem 5. *There exist simple non-hamiltonian* 3*-pyramids.*

Proof. Grinberg's graph shown in Fig. 24 is, by using Grinberg's Criterion, non-hamiltonian, and the 1-skeleton of a 3-pyramid, with basic cycles C_1, C_2 and C_3.

Further non-hamiltonian 3-pyramids appear in [1, items NH42.b and NH42.c in Fig. 2 and Fig. 8]. □

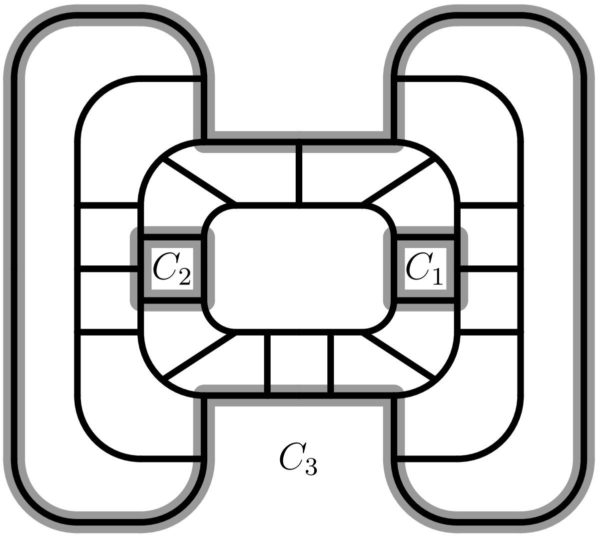


Fig. 24: A non-hamiltonian 3-pyramid with basic cycles highlighted.

4 Open Questions

Several natural questions arise. We select the following three.

Problem 1. Is every non-belted 3-pyramid hamiltonian?

Problem 2. Are all 3-pyramids traceable?

Problem 3. Let P be a simple polytope. P is certainly a k-pyramid for some number k. How can one efficiently determine the minimal k?

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