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Large Curvature on Typical Convex Surfaces*

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We show in this paper that on most convex surfaces there exist points with arbitrarily large lower curvature in every tangent direction.

Moreover, we show that, astonishingly, on most convex surfaces, although the set of points with curvature 0 in every tangent direction has full measure, it contains no pair of opposite points, i.e. points admitting parallel supporting planes.

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1. Introduction

The space $\mathcal{K}$ of all *convex bodies* (i.e. compact convex sets with non-empty interior) in $\mathbb{R}^d$, equipped with the Pompeiu-Hausdorff metric h, is a Baire space. Indeed, on one hand the space $\mathcal{C}$ of all convex compact sets is closed in the complete space of all compact sets (see e.g. [10]), hence itself complete, and on the other the lower-dimensional convex compact sets obviously form a nowhere dense set in $\mathcal{C}$.

The generic investigation of the smoothness (i.e. being of class C^1) and curvature of *convex surfaces* (i.e. boundaries of convex bodies) started with Victor Klee, who proved in 1959 the following result about *most* (or *typical*) convex bodies (i.e. all except those in a set of first Baire category).

Theorem 1.1 ([3]). *Most convex bodies are smooth and strictly convex.*

This was rediscovered later by Gruber [2], and refined in [6], [7], and [10], chapter 8.

Lemma Z. *If Z is a space of second Baire category, Y is residual in Z, and X is residual in Y, then X is residual in Z.*

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This is easy (or see [10], p. 17).

The first generic result about the curvature of convex surfaces was established by Peter Gruber [2] in 1977. He showed that most convex surfaces are not of class C^2. For arbitrary convex surfaces a lower and an upper curvature can be defined, as follows.

Consider a smooth, strictly convex body K, a point x on its boundary $\mathrm{bd}\, K$, and a tangent direction (unit vector) τ at x. Take the 2-dimensional half-plane

$$H = \{\alpha\tau + \beta\nu : \alpha > 0,\ \beta \in \mathbb{R}\}$$

spanned by τ and the outer normal unit vector ν of K in x. Then for any point $z \in (x + H) \cap \mathrm{bd}\, K$, there is exactly one circle with its centre on the normal line $x + \mathbb{R}\nu$, containing both x and z. Let r_z be the radius of this circle. Then

$$\rho_i^\tau(x) = \liminf_{z \to x} r_z, \qquad \rho_s^\tau(x) = \limsup_{z \to x} r_z$$

are called the *lower* and *upper curvature radius* and

$$\gamma_i^\tau(x) = \frac{1}{\rho_s^\tau(x)}, \qquad \gamma_s^\tau(x) = \frac{1}{\rho_i^\tau(x)}$$

the *lower* and *upper curvature* of $\mathrm{bd}\, K$ at x in direction τ. If $\gamma_i^\tau(x) = \gamma_s^\tau(x)$, then the *curvature in direction* τ exists and equals the common value $\gamma^\tau(x)$. (See [1], p. 14.)

Moreover, the intersection D of $x + \overline{H}$ with a ball containing x and having its centre in $x + \mathbb{R}\nu$ is called a *supporting half-disc* of K at x in direction τ if $D \subset K$.

In 1980 the second author proved the following.

Theorem 1.2 ([4]). *For most $K \in \mathcal{K}$, K is smooth and strictly convex; moreover, at each point $x \in \mathrm{bd}\, K$ and in any tangent direction τ at x,*

$$\gamma_i^\tau(x) = 0 \quad \textit{or} \quad \gamma_s^\tau(x) = \infty.$$

All theorems and corollaries of this paper refer – like Theorem 1.2 – to most convex bodies (or surfaces) and describe properties in addition to being smooth and strictly convex. From now on, in the statements we shall omit mentioning each time the smoothness and strict convexity, which are however necessary for the use of tangent directions, curvature etc.

Theorem 1.2, together with A. D. Alexandrov's well-known theorem which guarantees the existence of a finite curvature a.e. in all tangent directions, implies the following.

Corollary Y ([4]). *On most convex surfaces,*

$$\gamma^\tau(x) = 0 \ \text{ a.e.}$$

in all tangent directions τ at x.

Another result obtained on the curvature states the following.

Theorem 1.3 ([5]). *On most convex surfaces, at most points x,*

$$\gamma_i^\tau(x) = 0 \quad \text{and} \quad \gamma_s^\tau(x) = \infty$$

for any tangent direction τ.

The generic existence of points where a finite non-zero curvature in some tangent direction exists is excluded by Theorem 1.2. We saw that points with zero curvature must exist, by Corollary Y. The question about the generic existence of points with infinite curvature naturally arises. This could be proved for dimension $d = 2$ in both tangent directions [8]. In higher dimensions, only the existence of points with infinite curvature in some tangent directions could be shown [8], the following question remaining open.

Question 1.4. Does there exist, on most convex surfaces, a point where the curvature in every tangent direction is ∞? (1990, see [9])

Regarding all tangent directions, even the following question was unanswered.

Question 1.5. Do there exist, on most convex surfaces, points with arbitrarily large lower curvature in every tangent direction?

One goal of this paper is to answer Question 1.5.

2. Points of large lower curvature

Let $B(z, \alpha)$ denote the open ball of centre $z \in \mathbb{R}^d$ and radius $\alpha > 0$. Now, let us take a sphere S of radius $\epsilon > 0$, and a point x on that sphere. Let ν be the outer unit normal vector at x. Also, choose $\delta \in]0, 1[$. We define

$$S_\nu(x, \varepsilon, \delta) = \{y \in S : \rho(x, y) \leq \varepsilon\delta\} \tag{1}$$

where ρ is the inner metric on the sphere. Also define

$$C_\nu(x, \varepsilon, \delta) = \bigcup_{L \in \mathcal{K}, S_\nu(x,\varepsilon,\delta) \subset \operatorname{bd} L} L. \tag{2}$$

We write $S(x, \varepsilon, \delta)$ and $C(x, \varepsilon, \delta)$ instead of $S_\nu(x, \varepsilon, \delta)$ and $C_\nu(x, \varepsilon, \delta)$ if the vector ν is clear from the context.

If $K \in \mathcal{K}$, $x \in K$, and $K \subset C(x, \varepsilon, \delta)$, then we call $C(x, \varepsilon, \delta)$ a *hat* of K at x.

Hat Lemma. *Suppose K has a hat of radius $\varepsilon > 0$ at x. Then*

(i) *if K is smooth, $\rho_s^\tau(x) \leq \varepsilon$ for all tangent directions τ,*

(ii) *for any $\epsilon' > \epsilon$, there exists $\phi > 0$ so that each K' with $h(K, K') < \phi$ has a hat of radius ϵ' at some point in $B(x, \varepsilon')$.*

Proof. (i) is obvious.

We show (ii). Assume that, at x, $\operatorname{bd} K$ has the outer unit normal vector ν and K the hat $C(x,\varepsilon,\delta)$. We choose $\phi > 0$ smaller than $(1-\cos\delta)(\varepsilon'-\varepsilon)/3$. Consider a convex body K' with $h(K,K') < \phi$. Let $x' = x + (\varepsilon'-\varepsilon)\nu$ and

$$\lambda = \sup\{t : K' + t\nu \subset C(x',\varepsilon',\delta)\}.$$

Then

$$(K' + \lambda\nu) \cap \operatorname{bd} C(x',\varepsilon',\delta) \neq \emptyset.$$

Take y in the above set. We show that $y \in \operatorname{relint} S(x',\varepsilon',\delta)$. Indeed, suppose $y \in \operatorname{bd} C(x',\varepsilon',\delta) \setminus \operatorname{relint} S(x',\varepsilon',\delta)$. We have $y - \lambda\nu \in K'$, and $K' \subset C(x,\varepsilon,\delta) + B(\mathbf{0},\phi)$. Since the distance from y to $C(x,\varepsilon,\delta) + B(\mathbf{0},\phi)$ is $\varepsilon' - \varepsilon - \phi$, it follows that

$$\lambda > \frac{\varepsilon'-\varepsilon-\phi}{\cos\delta} > \frac{\varepsilon'-\varepsilon-\frac{(1-\cos\delta)(\varepsilon'-\varepsilon)}{3}}{\cos\delta} = \frac{\frac{2}{\cos\delta}+1}{3}(\varepsilon'-\varepsilon).$$

On the other hand, let x'' be the point of K' closest to x (possibly $x'' = x$), and ξ the number verifying $x'' + \xi\nu \in \operatorname{bd} C(x',\varepsilon',\delta)$. Of course, ξ should not be smaller than λ. But

$$\begin{aligned}\xi &\leq \|x'-x\| + \|x-x''\| \leq \varepsilon'-\varepsilon+\phi \\ &< \varepsilon'-\varepsilon+\frac{(1-\cos\delta)(\varepsilon'-\varepsilon)}{3} = \frac{4-\cos\delta}{3}(\varepsilon'-\varepsilon)\end{aligned}$$

and

$$\frac{4-\cos\delta}{3}(\varepsilon'-\varepsilon) < \frac{\frac{2}{\cos\delta}+1}{3}(\varepsilon'-\varepsilon),$$

because

$$\cos^2\delta - 3\cos\delta + 2 > 0,$$

which is true since $s^2 - 3s + 2 > 0$ for all $s \in]0,1[$. Hence, $\xi < \lambda$. This contradiction shows that $y \in \operatorname{relint} S(x',\varepsilon',\delta)$. Thus, K' admits a hat $C(y,\varepsilon',\delta')$ at y for some angle $\delta' > 0$. Here, $\|y-x\| < \max\{\|y-x'\|,\varepsilon'\} = \varepsilon'$, since $\|y-x'\| \leq \delta\varepsilon' < \varepsilon'$.

Proposition. Let $\varepsilon > 0$. The set of those convex bodies which have no hat of radius ε at any point of some open ball of radius ε centred at a boundary point of K is nowhere dense in $\mathcal{K}$.

Proof. Let $\mathcal{K}^*$ be the above set of convex bodies. In any open set of $\mathcal{K}$ we can find a polytope P such that every point on $\operatorname{bd} P$ has distance less than $\varepsilon/3$ from some vertex. Let $v_1, v_2, ..., v_n$ be the vertices of P. P admits a hat of radius $\varepsilon/4$ at every vertex. By the Hat Lemma (ii), we can find a neighbourhood $\mathcal{N}$ of P such that, for any $K \in \mathcal{N}$ with $h(K,K') < \varepsilon/3$ and for each index i, K admits a hat of radius $\varepsilon/3$ at some point at distance less than $\varepsilon/3$ from v_i. Now, for each $z \in \operatorname{bd} K$, there exists $z' \in \operatorname{bd} P$ with $\|z-z'\| < \varepsilon/3$, there exists some i for which $\|z'-v_i\| < \varepsilon/3$, and there exists a suitable hat of K at a point w with $\|w-v_i\| < \varepsilon/3$. Thus, $\|z-w\| < \varepsilon$, and $K \notin \mathcal{K}^*$.

Theorem 2.1. *For most convex bodies $K \in \mathcal{K}$ and for any number $r \in \mathrm{I\!R}$, there are densely many points $x \in \mathrm{bd}\, K$ such that*

$$\gamma_i^\tau(x) > r \quad \text{and} \quad \gamma_s^\tau(x) = \infty$$

in all tangent directions τ.

Proof. Let n be a natural number, and $\mathcal{K}_n$ be the family of all $K \in \mathcal{K}$ admitting no hat of radius $1/n$ at any point of some open ball of radius $1/n$ centred at a point of $\mathrm{bd}\, K$. By the Proposition, $\mathcal{K}_n$ is nowhere dense. Hence $\cup_{n=1}^{\infty} \mathcal{K}_n$ is of first category. Therefore, for most $K \in \mathcal{K}$, and for any natural number n, there are densely many points x at which K admits a hat of radius $1/n$. By the Hat Lemma (i), $\gamma_i^\tau(x) > n$ in all tangent directions τ. Moreover, by Theorem 1.2 (and Lemma Z), $\gamma_s^\tau(x) = \infty$.

3. Strengthening to nearly all convex bodies

Already results in [8] showed that for most convex bodies in $\mathrm{I\!R}^{\mathrm{d}}$, the lower curvature $\lambda_i^\tau(x)$, as a function of x and τ, is not bounded. By the Proposition, for any number $r \in \mathrm{I\!R}$, there is a point $x \in \mathrm{bd}\, K$ with $\gamma_i^\tau(x) > r$ in any tangent direction τ at x. We strengthen now this via porosity.

A subset $\mathcal{M}$ of $\mathcal{K}$ is said to be *porous* if, for any $K \in \mathcal{K}$, there exists $\alpha > 0$ such that, for any $\varepsilon > 0$, there exists $K' \in \mathcal{B}(K, \varepsilon)$ satisfying

$$\mathcal{M} \cap \mathcal{B}(K', \alpha h(K, K')) = \emptyset.$$

Here, $\mathcal{B}(K, \varepsilon) = \{K'' \in \mathcal{K} : h(K, K'') < \varepsilon\}$. We say that *nearly all* convex bodies have property $\mathbf{P}$ if those not enjoying $\mathbf{P}$ form a countable union of porous sets.

Theorem 3.1. *For nearly all convex bodies K and for any number $r \in \mathrm{I\!R}$, there is a point $x \in \mathrm{bd}\, K$ with $\gamma_i^\tau(x) > r$ in any tangent direction τ at x.*

Proof. Let us first recall that, in the terms of the Hat Lemma, the choice of

$$\phi = (1 - \cos\delta)(\varepsilon' - \varepsilon)/6 \tag{$*$}$$

suffices. Now let $\mathcal{A}_n$ be the set of all those smooth convex bodies with lower curvature in any point and in any tangent direction smaller than the natural number $n > 0$. Because nearly all convex bodies are smooth [6] (see also [10]), we can disregard all nonsmooth bodies. Thus, if we can prove that $\cup_{n \in \mathrm{I\!N}} \mathcal{A}_n$ is σ-porous, we are done. To achieve that, we will prove that every $\mathcal{A}_n$ is porous. Let us take $m > 0$ such that $K \in \mathcal{K}$ can be inscribed into an open ball of radius m. Obviously, there is a boundary point x of K such that K has a hat of radius m in x. Let δ be the angle of this hat $C_{\nu_x}(x, m, \delta)$, where ν_x is the outer normal of $\mathrm{bd}\, K$ in x. Let further θ be an arbitrary positive real number. We define $x_\theta = x + \theta\nu_x$,

$$K_\theta = \mathrm{conv}(\{x_\theta\} \cup K), \qquad C_\theta = \mathrm{conv}(\{x_\theta\} \cup C_{\nu_x}(x, m, \delta)).$$

In x_θ, both C_θ and K_θ have the hat

$$C_{\nu_x}\left(x_\theta, \frac{1}{2n}, \delta_\theta\right),$$

where

$$\delta_\theta = \min\left(\delta, \arccos\frac{m}{m+\theta}\right).$$

Now recall that if we choose

$$\phi(\theta) = \frac{1}{2n}\frac{1-\cos\delta_\theta}{6}$$

then, by the Hat Lemma, using $(*)$, every $K' \in \mathcal{B}(K_\theta, \phi(\theta))$ has a hat $C(y, 1/n, \delta')$ with some $\delta' > 0$. Now, for small θ,

$$1-\cos\delta_\theta = 1-\frac{m}{m+\theta} \geq \frac{\theta}{2m},$$

whence

$$\phi(\theta) \geq \frac{1}{24mn}\theta.$$

Thus, we can choose $\theta > 0$ so that eventually

$$\mathcal{A}_n \cap \mathcal{B}\left(K_\theta, \frac{1}{24mn}h(K, K_\theta)\right) = \mathcal{A}_n \cap \mathcal{B}\left(K_\theta, \frac{1}{24mn}\theta\right) = \emptyset,$$

and this shows that $\mathcal{A}_n$ is porous.

4. Upper curvature at opposite points

Let $\mathcal{S}$ be the (Baire) space of all convex surfaces. Take $K \in \mathcal{K}$ smooth and strictly convex. For any $x \in \operatorname{bd} K$, let x^* denote the unique point of $\operatorname{bd} K$ such that the hyperplanes tangent to $\operatorname{bd} K$ at x and x^* are parallel.

Theorem 4.1. *On most convex surfaces there exists no point x and no tangent direction τ such that*

$$\gamma_s^\tau(x) < \infty \quad \textit{and} \quad \gamma_s^\tau(x^*) < \infty.$$

Proof. Let $\mathcal{S}^*$ be the family of all boundaries of smooth strictly convex bodies. By Theorem 1.1, this family is residual in $\mathcal{S}$. Let $\mathcal{S}_\varepsilon$ be the family of all convex surfaces containing a point x such that at both points x and x^* there are supporting half-discs of radius ε in direction τ. It is routine to show that $\mathcal{S}_\varepsilon$ is closed in $\mathcal{S}^*$.

Now we shall show that $\mathcal{S}_\varepsilon$ is nowhere dense in $\mathcal{S}^*$. Let $S \in \mathcal{S}_\varepsilon$. Choose a neighbourhood $\mathcal{N}$ of S in $\mathcal{S}$. Let $P \in \mathcal{N}$ be a polytope in general position, which in our context should mean that no pair of faces of positive dimension are parallel. Neither P, nor its parallel body $P + B(\mathbf{0}, \varepsilon/3)$ belongs to $\mathcal{S}_\varepsilon$. But $P + B(\mathbf{0}, \varepsilon/3) \in \mathcal{N} \cap \mathcal{S}^*$. Hence $\mathcal{S}_\varepsilon$ is nowhere dense in $\mathcal{S}^*$.

Each smooth convex surface containing a point x and a tangent direction τ at x such that

$$\gamma_s^\tau(x) < \infty \quad \text{and} \quad \gamma_s^\tau(x^*) < \infty$$

lies in $\mathcal{S}_\varepsilon$ for some $\varepsilon > 0$. Thus, the family $\mathcal{S}_*$ of all such surfaces lies in $\cup_{i=1}^\infty \mathcal{S}_{1/n}$. As each $\mathcal{S}_{1/n}$ is nowhere dense, it follows that $\mathcal{S}_*$ is of first category in $\mathcal{S}^*$. The theorem now follows from Lemma Z.

Corollary V. *On most convex surfaces, for any point x and tangent direction τ, $\gamma^\tau(x) = 0$ implies $\gamma_s^\tau(x^*) = \infty$.*

Corollary W. *On most convex surfaces, for almost all x,*

$$\gamma^\tau(x) = 0 \quad \textit{and} \quad \gamma_s^\tau(x^*) = \infty$$

in all tangent directions τ at x.

Proof. Corollaries V and W immediately follow from Theorem 3.1, Corollary Y and Theorem 1.2.

At a first glance, one may wonder how the situation described in Corollary W may occur, while almost everywhere the curvature vanishes. Indeed, if the mapping $x \mapsto x^*$ were Lipschitz, then a contradiction would be obtained. But, as it follows, this mapping is, on most convex surfaces not Lipschitz a.e.

In contrast to this, the preceding mapping transforms residual sets into residual sets, so Theorem 1.3 implies that, on most convex surfaces, for most points x, there is no curvature in both x and x^*.

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