On Longest Paths in Triangular Lattice Graphs

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Abstract. In this note we present two graphs embeddable into the equilateral triangular lattice and satisfying Gallai's property that every vertex is missed by some longest path.
Key Words: Longest path, triangular lattice.

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A graph G is *Hamiltonian* if there exists a *Hamiltonian cycle* in G, i.e. a cycle which passes through every vertex of G. A *Hamiltonian path* of a graph G is a path in G that contains all vertices of G. A graph with a Hamiltonian path is called *traceable*. A non-Hamiltonian graph G such that $G - v$ is Hamiltonian for every vertex v is said to be *hypohamiltonian*. Petersen's graph is a well-known example of a hypohamiltonian graph.

Motivated by the existence of hypohamiltonian graphs and before the discovery of the analogously defined hypotraceable graphs, Tibor Gallai in 1966 raised the question about the existence of a graph with the property that every vertex is missed by some longest path. This property will be called *Gallai's property*. Gallai's question was answered by H. Walther [6] in 1969, who found a (planar) graph on 25 vertices enjoying Gallai's property. Later an example with just 12 vertices was found by H. Walther and H. Voss [7] and, independently, by T. Zamfirescu [9]. Very recently, G. Brinkmann and N. van Cleemput [1] have verified the minimality of this order 12, by using computers.

For planar graphs, the smallest example so far, with 17 vertices, was found by W. Schmitz [4] in 1975.

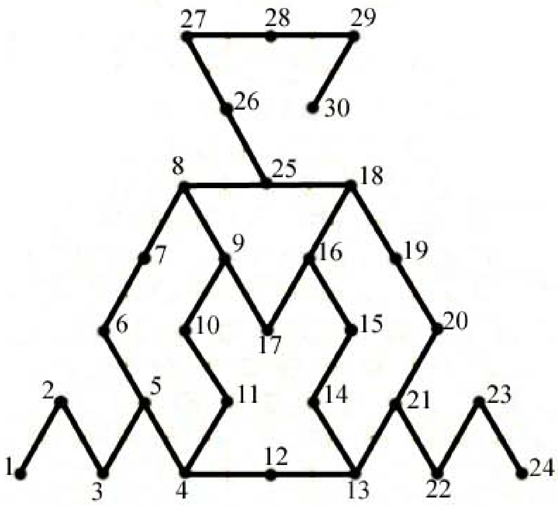

Figure 1

The first 2-connected graph constructed in 1972 by Zamfirescu [8] had 82 vertices and was planar. The smallest example known today has 26 vertices [10], while the smallest planar example so far has order 32 [9]. If we impose further restrictions on the graphs under consideration, Gallai's question may get a negative answer. If we ask for example that the graphs be planar and 4-connected, all vertices lie on every longest path, as these graphs are by a well-known result of Tutte Hamiltonian [5]. B. Menke, Ch. Zamfirescu and T. Zamfirescu [3] investigated a certain family of subgraphs of the square lattice in the plane, and concluded that, in this case too, no graph has Gallai's property.

In this note we ask the graph to be embeddable into the equilateral triangular lattice of the plane. We find out that, under this restriction, there exist graphs enjoying Gallai's property.

Triangular lattice graphs are a well-studied class of graphs with applications in areas like molecular biology (protein folding), configurational statistics of polymers, telecommunications, computer vision, or designing of cellular networks. For references, see for instance [2].

Theorem 1. *There exists a triangular lattice graph on* 30 *vertices satisfying Gallai's property.*

Proof. We use the graph in Figure 1, which is homeomorphic to the graph Z of Figure 1 in [9]. We now check that our graph inherits the

	Vertices Missed	Longest Path
1	$\underline{1}$, $\underline{2}$, $\underline{3}$, $\underline{5}$, $\underline{6}$, $\underline{7}$, $\underline{17}$	30, 29, 28, 27, 26, 25, 8, 9, 10, 11, 4, 12, 13, 14, 15, 16, 18, 19, 20, 21, 22, 23, 24
2	$\underline{10}$, $\underline{11}$, $\underline{12}$, $\underline{4}$, $\underline{22}$, $\underline{23}$, $\underline{24}$	1, 2, 3, 5, 6, 7, 8, 9, 17, 16, 15, 14, 13, 21, 20, 19, 18, 25, 26, 27, 28, 29, 30
3	6, 7, $\underline{8}$, 22, 23, 24, 12	1, 2, 3, 5, 4, 11, 10, 9, 17, 16, 15, 14, 13, 21, 20, 19, 18, 25, 26, 27, 28, 29, 30
4	1, 2, 3, $\underline{9}$, 10, 11, 17	24, 23, 22, 21, 20, 19, 18, 16, 15, 14, 13, 12, 4, 5, 6, 7, 8, 25, 26, 27, 28, 29, 30
5	1, 2, 3, 12, $\underline{13}$, $\underline{14}$, $\underline{15}$	24, 23, 22, 21, 20, 19, 18, 16, 17, 9, 10, 11, 4, 5, 6, 7, 8, 25, 26, 27, 28, 29, 30
6	14, 15, $\underline{16}$, 17, 22, 23, 24	1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 4, 12, 13, 21, 20, 19, 18, 25, 26, 27, 28, 29, 30
7	1, 2, 3, 12, $\underline{18}$, $\underline{19}$, $\underline{20}$	24, 23, 22, 21, 13, 14, 15, 16, 17, 9, 10, 11, 4, 5, 6, 7, 8, 25, 26, 27, 28, 29, 30
8	17, 19, 20, $\underline{21}$, 22, 23, 24	1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 4, 12, 13, 14, 15, 16, 18, 25, 26, 27, 28, 29, 30
9	17, $\underline{25}$, $\underline{26}$, $\underline{27}$, $\underline{28}$, $\underline{29}$, $\underline{30}$	1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 4, 12, 13, 14, 15, 16, 18, 19, 20, 21, 22, 23, 24

property of Z of having empty intersection of all its longest paths. The table shows 9 longest paths and all vertices missed by each of them. The underlined vertices are from 1 to 30, so each vertex is avoided by some longest path.

Since the above graph is not 2-connected, one may ask if 2-connected examples can be found.

Theorem 2. *There exists a* 2*-connected triangular lattice graph on* 92 *vertices satisfying Gallai's property.*

Proof. This time we use the graph in Figure 2, which is homeomorphic to the graph of Figure 4 in [9]. It can be checked that our graph inherits Gallai's property from that graph.

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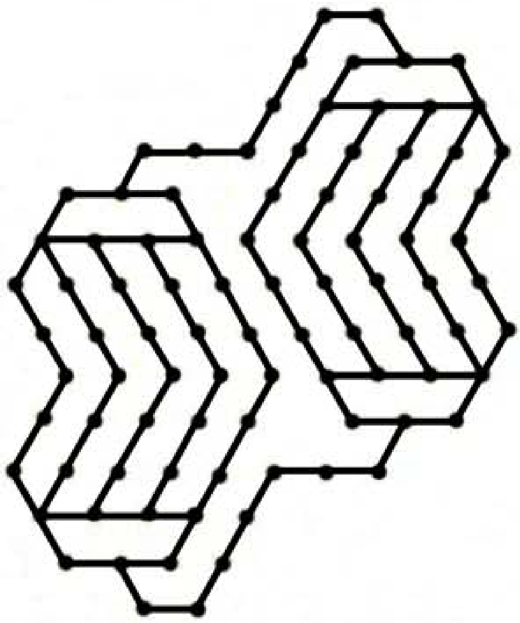

Figure 2

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