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ORIGINAL PAPER

On Planar Toeplitz Graphs

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Abstract We describe several classes of finite, planar Toeplitz graphs and present results on their chromatic number. We then turn to counting maximal independent sets in these graphs and determine recurrence equations and generating functions for some special cases.

Keywords Toeplitz graph · Planarity · Colouring · Maximal independent set · Counting

Mathematics Subject Classification 05C75 · 05C15 · 05A15

1 Introduction

Let $T = (V, E)$ be an undirected, simple graph with $V = \{1, \ldots, n\}$. We call T *Toeplitz* if its adjacency matrix $A(T)$ is Toeplitz, i.e. identical on all its diagonals parallel to the main diagonal. A Toeplitz graph T is therefore uniquely defined by the first row of $A(T)$, a $(0-1)$-sequence. If the 1's in that sequence are placed at positions

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$1+t_1, \ldots, 1+t_k$ with $0 < t_1 < \cdots < t_k < n$, we may simply write $T = T_n\langle t_1, \ldots, t_k\rangle$, two vertices x, y of T being connected by an edge iff $|x-y| \in \{t_1, \ldots, t_k\}$. For $V = \mathbb{N}$ and $k < \infty$ *infinite* Toeplitz graphs $T = T_\infty\langle t_1, \ldots, t_k\rangle$ are defined the same way. We simply mention that both types may be studied as special subgraphs of integer distance graphs.

Toeplitz graphs have been introduced by G. Sierksma and first been investigated with respect to hamiltonicity by van Dal et al. [2] (see also Heuberger [8], Malik and Qureshi [11], Malik and Zamfirescu [12] for more recent work). Infinite, bipartite Toeplitz graphs have been fully characterized in terms of bases and circuits by Euler et al. [6] (with results on the finite case presented in Euler [3]). Colouring aspects are especially treated in Heuberger [9], Kemnitz and Marangio [10], Nicoloso and Pietropaoli [14]. Infinite, planar Toeplitz graphs, finally, have been fully characterized in Euler [4] providing, in particular, a complete description of the class of 3-colourable such graphs.

This paper is organized as follows: in Sect. 2 we present several classes of finite, planar Toeplitz graphs, Sect. 3 is on colouring aspects, and Sect. 4 is devoted to counting maximal independent sets in special instances of these graphs. We just mention that counting such sets in planar graphs has been shown by Vadhan [16] to be #$\mathcal{P}$-complete.

2 Finite, Planar Toeplitz Graphs

Let us start by recalling the infinite case, which was investigated in [4]. For that case an infinite sequence $(a_n)_{n\in\mathbb{N}}$ is said to *dominate* a sequence $(b_n)_{n\in\mathbb{N}}$ if $a_i \geq b_i$ for all $i \in \mathbb{N}$.

Theorem 1 *An infinite* $(0-1)$*-sequence S defines a planar Toeplitz graph if and only if S is dominated by a* $(0-1)$*-sequence whose* 1*-entries are at positions* $1+t_1$, $1+t_2$ *and* $1+(t_1+t_2)$.

Consequently, for infinite, planar Toeplitz graphs $T = T_\infty\langle t_1, \ldots, t_k\rangle$, k can be no more than 3. Under the circumstances treated in the following, we will show that k must remain rather small if planarity is required. This is not, however, a general rule. In fact, the next result shows that k can be arbitrarily large and planarity still preserved.

Theorem 2 *If* $T = T_n\langle t_1, \ldots, t_k\rangle$ *is planar and* $c \in \mathbb{N}$*, then* $T_{cn}\langle ct_1, \ldots, ct_k, cn-1\rangle$ *is planar, too.*

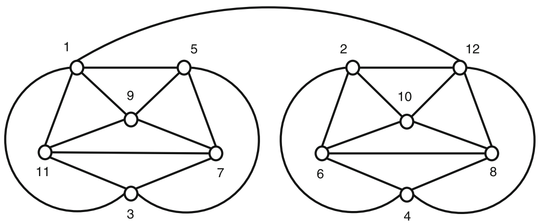
Proof Observe that $T_{cn}\langle ct_1, \ldots, ct_k\rangle$ has c pairwise disjoint subgraphs, each of them isomorphic to $T_n\langle t_1, \ldots, t_k\rangle$: the adjacency matrix of T simply decomposes into c identical $(0-1)$-matrices. We embed the subgraphs in the plane in such a way that the vertex 1 of the first and the vertex n of the last appear on the boundary of the unbounded region. Thus, adding the edge $\{1, cn\}$ does preserve planarity. In Fig. 1 we show a planar Toeplitz graph with $k = 5$. □

As an immediate consequence we obtain

Corollary 1 $T_{t_1+t_2}\langle t_1, t_2, t_1+t_2-1\rangle$ *is planar.*

To see this we just recall that the graph $T_{t_1+t_2}\langle t_1, t_2\rangle$ is planar, being a cycle or a union of pairwise disjoint cycles; hence Theorem 2 applies. □

Fig. 1 The Toeplitz graph $T_{12}\langle 2, 4, 8, 10, 11\rangle$



Since planarity is hereditary, it follows from Theorem 1 that the finite graph $T_n\langle t_1, t_2, t_1 + t_2\rangle$ and its Toeplitz subgraphs $T_n\langle t_1, t_2\rangle$, $T_n\langle t_1\rangle$ are planar for all t_1, t_2 and all $n \geq t_1+t_2+1, t_2+1, t_1+1$, respectively. To make this paper self-contained we will give a separate proof of this result later. Also note that the finite case differs from the infinite one with respect to connectivity: in general, a finite Toeplitz graph only decomposes into *at least* $c = gcd(t_1, \ldots, t_k)$ many connected (and not necessarily isomorphic) components (cf. [2]), whereas c is the *exact* number of connected and isomorphic components in the infinite case (cf. [4]). For convenience, we will restrict ourselves to the case $gcd(t_1, t_2) = 1$.

In view of Theorem 1, we may consider a finite Toeplitz graph $T_n\langle t_1, \ldots, t_k\rangle$ with $k \geq 3$ which is not the graph $T_n\langle t_1, t_2, t_1 + t_2\rangle$ and ask the question, from which n on planarity will be lost. It turns out that this is the case whenever $n \geq 2t_1 + 2t_2 - 1$. Three cases arise:

(i) $n \geq 2t_1 + 2t_2 - 1$
(ii) $n \leq 2t_1$
(iii) $2t_1 < n < 2t_1 + 2t_2 - 1$.

For the third case, an example is given by $T_{t_1+t_2}\langle t_1, t_2, t_1 + t_2 - 1\rangle$, and we think that investigating this situation further should be an exciting future task.

2.1 The Case $n \geq 2t_1 + 2t_2 - 1$

We will show that for $k \geq 3$ and $t_2 > 2$ planarity implies $k \leq 3$ and $t_3 = t_1 + t_2$. For this the following proof of the planarity of $T_n\langle t_1, t_2\rangle$ will be very useful:
We embed $T = T_n\langle t_1, t_2\rangle$ in the plane by using the infinite planar square lattice graph $\mathcal{L}$ as follows: for vertex 1 we choose arbitrarily some lattice point. Then we label the points below with $1 + t_1, 1 + 2t_1, \ldots$, and with $1 + t_2, 1 + 2t_2, \ldots$ those to the right of 1. Further, we take any of the points $1 + it_2$ and label the points below with $1 + it_2 + t_1, 1 + it_2 + 2t_1, \ldots$ (if any). For each j we complete the finite sequence $1 + jt_1, 1 + t_2 + jt_1, 1 + 2t_2 + jt_1, \ldots$ to the left with $1 + jt_1 - t_2, 1 + jt_1 - 2t_2, \ldots$. These numbers, being vertices of $T_n\langle t_1, t_2\rangle$, all lie between 1 and n. For $j = t_2$, the horizontal finite sequence becomes again $1, 1 + t_2, 1 + 2t_2, \ldots$, and the procedure continues. In this way we get an infinite subgraph $\mathcal{H}$ of $\mathcal{L}$. By identifying all points of $\mathcal{H}$ carrying the same number we obtain a graph $\mathcal{G}$ which is both planar and isomorphic to T.

Lemma 1 *$T_n\langle t_1, t_2, t_1 + t_2\rangle$ is planar.*

Proof We use the previous embedding of $T_n\langle t_1, t_2\rangle$ in the plane and the graph $\mathcal{H}$. Indeed, we immediately recognize that any edge $\{x, x + t_1 + t_2\}$ can be added to $\mathcal{H}$ as a diagonal in the lattice square induced by $x, x + t_1, x + t_1 + t_2, x + t_2$. Planarity is not violated either when passing from $\mathcal{H}$ to $\mathcal{G}$. □

We will see in the following that planarity of Toeplitz graphs is in an intimate and not quite obvious relationship with the connectivity of $\mathcal{H}$. Let $\mathcal{H}^*$ be the infinite graph homeomorphic to $\mathcal{H}$ and of smallest degree 3. (Thus, vertices such as 1 will disappear.) Similarly, let $\mathcal{G}^*$ be the graph homeomorphic to $\mathcal{G}$ and of smallest degree 3.

Lemma 2 *If $c \leq 4$ and $n \geq c(t_1 + t_2)$, then $\mathcal{H}^*$ is c-connected.*

The easy proof is left to the reader.

Lemma 3 *If $c \leq 2$ and $\mathcal{H}^*$ is c-connected, then $\mathcal{G}^*$ is $2c$-connected.*

Proof Indeed, we must remove two disjoint cut sets from $\mathcal{H}^*$ or a 4-vertex set (the neighbourhood of some vertex) to disconnect $\mathcal{G}^*$. □

Lemma 4 *Suppose the connectivity of $\mathcal{H}$ is 1 and all its cutpoints carry the same number (become the same vertex of $\mathcal{G}$ after identification). Then $\mathcal{G}^*$ is 3-connected.*

Proof Any cut set of $\mathcal{G}^*$ with less than 4 vertices is the union of two disjoint cut sets of $\mathcal{H}^*$ numbered differently. Since only one of them can consist of a single vertex, any cut set of $\mathcal{G}^*$ has at least 3 vertices. □

We are now able to show

Theorem 3 *If $gcd(t_1, t_2) = 1, t_2 > 2, k \geq 3$ and $n \geq 2t_1 + 2t_2 - 1$ for a planar Toeplitz graph $T = T_n\langle t_1, t_2, \ldots, t_k\rangle$, then $T = T_n\langle t_1, t_2, t_1 + t_2\rangle$.*

Proof We use the above terminology and consider $T_n\langle t_1, t_2\rangle$ first. If n equals $(2t_1 + 2t_2 - 1)$ and t_2 is odd, we are led precisely to the situation of Lemma 3; otherwise we get $\mathcal{H}$ and $\mathcal{H}^*$ 2-connected, which implies by Lemma 2 the 4-connectivity of $\mathcal{G}^*$. Thus, $\mathcal{G}^*$ is a polytopal graph, whence the regions in which $\mathcal{G}^*$ divides the plane are uniquely determined. Some vertices of $\mathcal{G}$, like 1, are not vertices of $\mathcal{G}^*$, but the homeomorphism between $\mathcal{G}$ and $\mathcal{G}^*$ suggests to say that they belong to certain edges of $\mathcal{G}^*$ (so 1 belongs to the edge $\{1 + t_1, 1 + t_2\}$ of $\mathcal{G}^*$). In this way all vertices $1, 2, \ldots, t_1 + t_2$ belong to the boundary of the same region and no other vertex of $\mathcal{G}^*$ lies on that boundary. Thus, the only new edges incident at $t_1 + t_2 - 1$, which could be added preserving planarity, have the other vertex on the boundary of one of the two incident regions, i.e., in the set $\{1, 2, \ldots, t_1 + t_2, 2t_1 + t_2 - 1, t_1 + 2t_2 - 1, 2t_1 + 2t_2 - 1\}$. An edge produced by t_3 is $\{t_1 + t_2 - 1, t_1 + t_2 + t_3 - 1\}$. Since $t_3 > t_2$, the only possibility is $t_1 + t_2 + t_3 - 1 = 2t_1 + 2t_2 - 1$, that is $t_3 = t_1 + t_2$. □

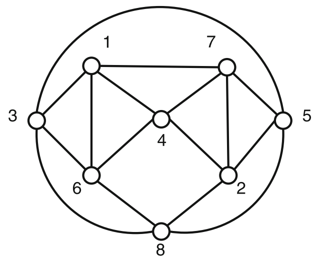


Fig. 2 The Toeplitz graph $T_8\langle 2, 3, 5, 6\rangle$

The lower bound on n given in Theorem 3 is best possible. Figure 2 shows the planar Toeplitz graph $T_8\langle 2, 3, 5, 6\rangle$. By Theorem 3 the only planar Toeplitz graph with $t_1 = 2, t_2 = 3, k \geq 3$ and $n \geq 9$ is $T_n\langle 2, 3, 5\rangle$.

What can we say if $gcd(t_1, t_2, t_3) = 1$?

If $gcd(t_1, t_2) = c > 1$ and $c|t_3$ then the preceding Theorem gives important information on the planarity of the c components of $T_n\langle t_1, t_2, t_3\rangle$. This, in turn, can be used for every Toeplitz graph admitting $T_n\langle t_1, t_2, t_3\rangle$ as a subgraph. However, the case $gcd(t_1, t_2, t_3) = 1$ appears to be untractable with our present knowledge if $gcd(t_i, t_j) > 1$ for $i, j \in \{1, 2, 3\}, i \neq j$.

Still we have the following result.

Theorem 4 *Let p, q, r be primes and $n > pqr$. Then $T_n\langle pq, pr, qr\rangle$ is not planar.*

Proof We may assume $p < q < r$. First, suppose $\{p, q, r\} \neq \{2, 3, 5\}$. Then $r \geq 7$. We find the following subgraph of $T_n\langle pq, pr, qr\rangle$ homeomorphic to $K_{3,3}$: one set of vertices is $\{pq+1, pr+1, qr+1\}$, the other $\{1, p(q+r)+1, pqr+1\}$. Obviously, 1 is adjacent to $pq+1$, $pr+1$ and $qr+1$. Vertex $pqr+1$ is joined with $pq+1$ by the path $[pqr+1, pq(r-1)+1, \ldots, pq+1]$, with $pr+1$ by the path $[pqr+1, p(q-1)r+1, \ldots, pr+1]$ and with $qr+1$ by the path $[pqr+1, (p-1)qr+1, \ldots, qr+1]$. These paths are pairwise disjoint (except at $pqr+1$), because if $pqi+1 = pjr+1$, say, then $q|j$, which contradicts $j < q$. Finally, the vertex $p(q+r)+1$ is adjacent to $pr+1$ and to $pq+1$. It remains to show that it is also joined to $qr+1$ by a suitable path. Indeed, the path $P = [p(q+r)+1, pq+pr+qr+1, pr+qr+1, qr+1]$ does not meet any previous path (except at $p(q+r)+1$ and $qr+1$), because if $pq+pr+qr+1$ or $pr+qr+1$ equals $pqi+1$ or $prj+1$ or $qrl+1$, then $p|q$ or $p|r$, a contradiction. Also, we have to verify that P is entirely contained in $T_n\langle pq, pr, qr\rangle$, i.e., that

$$pq + pr + qr + 1 \leq n.$$

Actually, if $p \geq 3$ the strict inequality holds:

$$pq + pr + qr + 1 < 3qr + 1 \leq pqr + 1 \leq n.$$

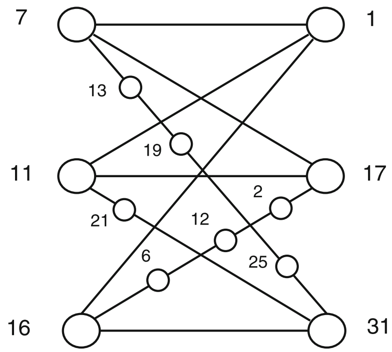


Fig. 3 A non-planar subgraph for the case $\{p, q, r\} = \{2, 3, 5\}$

For $p = 2$ we first see that $q \geq 3$ and $r \geq 7$ imply

$$2q \leq 6(q-2) < r(q-2),$$

whence $2(q+r) < qr$ and

$$2q + 2r + qr + 1 < 2qr + 1 \leq n.$$

It remains to treat the particular case $\{p, q, r\} = \{2, 3, 5\}$. Figure 3 shows a subgraph homeomorphic to $K_{3,3}$ in $T_{31}\langle 6, 10, 15\rangle$. □

2.2 The Case $n \leq 2t_1$

In case that $t_2 = t_1 + 1$ we are able to find all planar Toeplitz graphs. In particular, $k \leq 4$ must hold. If $t_2 > t_1 + 1$ we point out the existence of planar Toeplitz graphs with $k = 5$.

Theorem 5 *If for a planar Toeplitz graph T, $n \leq 2t_1$ and $t_2 = t_1 + 1$, then T is $T_n\langle t_1, t_1 + 1, t_3, t_3 + 1\rangle$ or a Toeplitz subgraph of it.*

Proof Figure 4 shows a planar embedding of $T_n\langle t_1, t_1 + 1, t_3, t_3 + 1\rangle$ for $t_3 = t_1 + 4$, the generalization to arbitrary t_3 being straightforward.

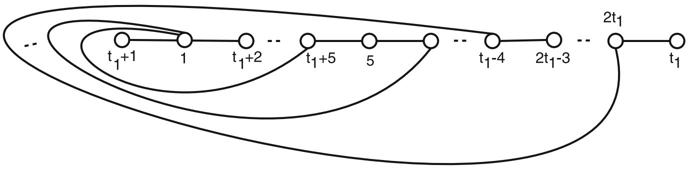


Fig. 4 The Toeplitz graph $T_{2t_1}\langle t_1, t_1 + 1, t_1 + 4, t_1 + 5\rangle$

Consider now $T_n\langle t_1, t_1+1, t_3, t_4\rangle$ with $t_4 \geq t_3+2$. We show the existence of a subgraph homeomorphic to $K_{3,3}$. The point sets will be $\{1, 3, t_3+2\}$ and $\{2, t_3+1, t_3+3\}$. Now, 1 is joined to 2 by the path $[1, t_1+2, 2]$, is adjacent to t_3+1, and either adjacent to t_3+3 if $t_4 = t_3+2$ or joined to t_3+3 by the path $[1, t_4+1, t_4-t_1, t_4, \ldots, t_3-t_1+3, t_3+3]$ otherwise. Also, 3 is joined to 2 by the path $[3, t_1+3, 2]$ and to t_3+1 by the path $[3, t_1+4, 4, \ldots, t_3-t_1, t_3+1]$ and is adjacent to t_3+3. Finally, t_3+2 is adjacent to 2 and joined by the paths $[t_3+2, t_3-t_1+1, t_3+1]$ and $[t_3+2, t_3-t_1+2, t_3+3]$ with t_3+1 and t_3+3. □

Theorem 6 *Suppose* $t_2 > t_1+1$ *and* $n \leq 2t_1$. *If* $(t_2-t_1)|(t_3-t_1)$ *then* $T_n\langle t_1, t_2, t_3, t_2+t_3-t_1, n-1\rangle$ *is planar.*

Proof The graph $T_n\langle t_1, t_2\rangle$ has t_2-t_1 components. Each of them reduces to the case $T_n\langle t_1, t_1+1\rangle$, which was treated in Theorem 5. In the (t_2-t_1)-th component, t_2-t_1 is adjacent to t_2 and to $2t_2-t_1$, vertex $2(t_2-t_1)$ is adjacent to $2t_2-t_1$ and to $3t_2-2t_1$, etc. Thus, a new series of edges, produced by another diagonal in the adjacency matrix and preserving planarity, joins t_2-t_1 to any $jt_2-(j-1)t_1$ and corresponds to $t_3 = (j-1)t_2-(j-2)t_1$. This means that $t_3-t_1 = (j-1)(t_2-t_1)$ and happens precisely when $(t_2-t_1)|(t_3-t_1)$. By applying Theorem 5 we find then that planarity is also kept by joining t_2-t_1 to $(j+1)t_2-jt_1$ (in fact, each $i(t_2-t_1)$ to $(j+i)t_2-(j+i-1)t_1$), which corresponds to

$$t_4 = jt_2-(j-1)t_1 = t_2+t_3-t_1.$$

Finally, the first component can be embedded in the plane so that vertex 1 appears on the boundary of its unbounded region. The (t_2-t_1)-th component can in turn be embedded so that vertex n appears on the boundary of its unbounded region. Then clearly, $t_5 = n-1$ yields a planar graph (disconnected, provided $t_2-t_1 > 2$). □

2.3 All Finite, Planar Toeplitz Graphs with $t_1 = 1$

We know already that $T_n\langle 1, t_2, t_2+1\rangle$ is the only planar Toeplitz graph with $t_1 = 1$ and $t_2 > 2$ in case that $n \geq 2t_2+1$. That this remains true for any n follows from the next lemma.

Lemma 5 $T_{t_3+1}\langle 1, t_2, t_3\rangle$ *is not planar for* $t_2 > 2$ *and* $t_3 \geq t_2+2$.

Proof First assume that $t_3 < 2t_2$ (see Fig. 5). Then $t_3+1-t_2 < t_2+1$. We find a subgraph homeomorphic to $K_{3,3}$ in T as follows: consider the two vertex sets $\{1, t_3+1-t_2, t_2+2\}$ and $\{2, t_2+1, t_3+1\}$. T contains the edges $\{1, 2\}$, $\{1, t_2+1\}$ and $\{1, t_3+1\}$. The vertex t_3+1-t_2 is joined by the path $[t_3+1-t_2, t_3-t_2, \ldots, 3, 2]$ to 2, by the path $[t_3+1-t_2, t_3+2-t_2, \ldots, t_2, t_2+1]$ to t_2+1, and adjacent to t_3+1. Finally, t_2+2 is adjacent to 2 and to t_2+1, and joined by the path $[t_2+2, t_2+3, \ldots, t_3, t_3+1]$ to t_3+1.

Now suppose that $t_3 \geq 2t_2$. Then $t_3-2t_2+2 \geq 2$. Again, we find a subgraph homeomorphic to $K_{3,3}$ in T: the two vertex sets are $\{t_3-2t_2+2, t_3-t_2+1, t_3\}$ and $\{t_3-t_2, t_3-t_2+2, t_3+1\}$. Indeed, t_3-2t_2+2 is joined to t_3-t_2 by the path

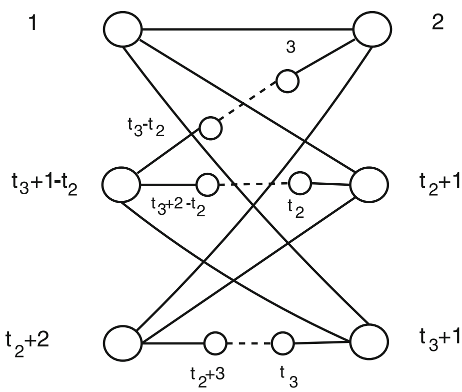


Fig. 5 A non-planar subgraph for the case $t_3 < 2t_2$

$[t_3 - 2t_2 + 2, t_3 - 2t_2 + 3, \ldots, t_3 - t_2]$, is adjacent to $t_3 - t_2 + 2$, and is joined to $t_3 + 1$ by the path $[t_3 - 2t_2 + 2, t_3 - 2t_2 + 1, \ldots, 2, 1, t_3 + 1]$. The vertex $t_3 - t_2 + 1$ is adjacent to each of the vertices $t_3 - t_2$, $t_3 - t_2 + 2$ and $t_3 + 1$. Finally, t_3 is adjacent to $t_3 - t_2$, joined by the path $[t_3, t_3 - 1, \ldots, t_3 - t_2 + 3, t_3 - t_2 + 2]$ to $t_3 - t_2 + 2$, and adjacent to $t_3 + 1$. □

Lemma 6 *$T_{t_3+3}\langle 1, 2, t_3\rangle$ is not planar for $t_3 \geq 4$.*

Proof Consider the vertex sets $\{1, 4, t_3+2\}$ and $\{2, 3, t_3+1\}$. Vertex 1 is adjacent to 2, 3 and t_3+1. Vertex 4 is adjacent to 2 and 3, and joined by the path $[4, 5, \ldots, t_3, t_3+1]$ to $t_3 + 1$. Vertex $t_3 + 2$ is adjacent to 2, joined by the path $[t_3 + 2, t_3 + 3, 3]$ to 3, and adjacent to $t_3 + 1$. □

Lemma 7 *$T_{t_3+2}\langle 1, 2, t_3\rangle$ is not planar for any odd $t_3 \geq 5$.*

Proof Take the same vertex sets $\{1, 4, t_3 + 2\}$ and $\{2, 3, t_3 + 1\}$ as in the preceding proof, and the same paths with two exceptions: the path joining 4 to $t_3 + 1$ will now be $[4, 6, \ldots, t_3-1, t_3+1]$ and the path joining t_3+2 to 3 will now be $[t_3+2, t_3, \ldots, 5, 3]$. □

Lemma 8 *$T_{t_3+2}\langle 1, 2, t_3, t_3 + 1\rangle$ is planar for t_3 even.*

Proof Figure 6 presents a planar embedding of $T_{t_3+2}\langle 1, 2, t_3, t_3 + 1\rangle$. □

Altogether, we obtain

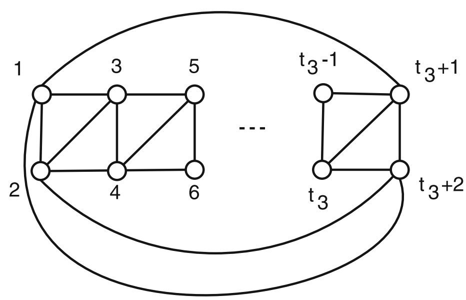


Fig. 6 The Toeplitz graph $T_{t_3+2}\langle 1, 2, t_3, t_3 + 1\rangle$

Theorem 7 *The finite, planar Toeplitz graphs with* $t_1 = 1$ *are* $T_n\langle 1, t_2, t_2 + 1\rangle$ *and* $T_{t_3+2}\langle 1, 2, t_3, t_3 + 1\rangle$ *with* t_3 *even, plus all their Toeplitz subgraphs* (*with* $t_1 = 1$).

Proof By Lemmas 1 and 8, the graphs from the statement are planar. By Lemma 5, if a Toeplitz graph with $t_1 = 1$ and $k = 3$ is planar and different from $T_n\langle 1, t_2, t_2 + 1\rangle$, then $t_2 = 2$. In this case, by Lemma 6, $n \leq t_3 + 2$. Moreover, Lemma 7 forces t_3 to be even. Obviously, $k \leq 4$. □

Corollary 2 $T_{t_3+1}\langle 1, 2, t_3\rangle$ *is planar.*

3 Colouring Aspects

It is well known (see for instance [1]) that any infinite Toeplitz graph $T = T_\infty\langle t_1, \ldots, t_k\rangle$ can be coloured with $k + 1$ colours by a greedy-like algorithm. Hence, planar such graphs and, in particular, the graphs $T_n\langle t_1, t_2, t_1 + t_2\rangle$, $T_n\langle t_1, t_2\rangle$, $T_n\langle t_1\rangle$ are immediately seen to be 4-colourable. We are interested in the *chromatic number* $\chi(T)$ of a finite, planar Toeplitz graph T, i.e., the minimum number p for which T has a p-colouring. The aim of this section is to determine this number for all those families of Toeplitz graphs that have been presented in Sect. 2.

For the infinite case we have the following result.

Theorem 8 [4] *Let* 2^r *and* 3^s *be the highest powers of 2 and 3 that divide* t_1.

(i) $T = T_\infty\langle t_1\rangle$ *is always bipartite.*
(ii) *If* $T = T_\infty\langle t_1, t_2\rangle$, *then*

$$\chi(T) = \begin{cases} 2 & \text{if } 2^{r+1} | (t_2 - t_1) \\ 3 & \text{if not.} \end{cases}$$

(iii) *If* $T = T_\infty\langle t_1, t_2, t_1 + t_2\rangle$, *then*

$$\chi(T) = \begin{cases} 3 & \text{if } 3^{s+1} | (t_2 - t_1) \\ 4 & \text{if not.} \end{cases}$$

In the finite case, $T_n\langle t_1\rangle$ is always bipartite, and for $k = 2$ we have

Lemma 9 *For* $T = T_n\langle t_1, t_2\rangle$,

(i) *if* $2^{r+1} | (t_2 - t_1)$, *then* T *is bipartite for any* $n \in \mathbb{N}$;

(ii) *if* $2^{r+1} \nmid (t_2 - t_1)$, *then*

$$\chi(T) = \begin{cases} 2 & \text{if } n \leq t_1 + t_2 - gcd(t_1, t_2), \\ 3 & \text{if } n > t_1 + t_2 - gcd(t_1, t_2). \end{cases}$$

Proof (i) follows directly from Theorem 8. For (ii), let $c := gcd(t_1, t_2)$ and $t_i' := t_i/c$ for $i = 1, 2$. We know that $T\langle t_1, t_2\rangle$ decomposes into c isomorphic components. Suppose now that $t_1' + t_2'$ is even. Then both of t_1' and t_2' have to be odd and 2 divides $(t_2' - t_1')$. But this means that $T\langle t_1, t_2\rangle$ is bipartite, a contradiction. The finite Toeplitz graph $T_{t_1+t_2}\langle t_1, t_2\rangle$ thus decomposes into c cycles of odd length $t_1' + t_2'$. Since the c vertices $t_1 + t_2, t_1 + t_2 - 1, \ldots, t_1 + t_2 - c + 1$ all belong to different components of $T_{t_1+t_2}\langle t_1, t_2\rangle$, the maximum number n^* for which $T_{n^*}\langle t_1, t_2\rangle$ is bipartite, is given by $n^* = t_1 + t_2 - gcd(t_1, t_2)$. □

A corresponding result for the case $k = 3$ and $t_3 = t_1 + t_2$ is as follows.

Lemma 10 *For* $T = T_n\langle t_1, t_2, t_1 + t_2\rangle$,

(i) *if* $2^{r+1} \mid (t_2 - t_1)$, *then*

$$\chi(T) \begin{cases} = 2 & \text{if } n \le t_1 + t_2, \\ > 2 & \text{if } n > t_1 + t_2. \end{cases}$$

(ii) *if* $2^{r+1} \not| \, (t_2 - t_1)$, *then*

$$\chi(T) \begin{cases} = 2 & \text{if } n \le t_1 + t_2 - gcd(t_1, t_2), \\ > 2 & \text{if } n > t_1 + t_2 - gcd(t_1, t_2). \end{cases}$$

For a proof, we just observe that

$$T_n\langle t_1, t_2, t_1 + t_2\rangle \begin{cases} \text{– coincides with } T_n\langle t_1, t_2\rangle, \text{ whenever } n \le t_1 + t_2, \\ \text{– contains a triangle induced by } \{1, 1 + t_1', 1 + t_1' + t_2'\}, \\ \qquad \text{if } n > t_1 + t_2. \end{cases}$$

It remains to determine the maximum number n^* for which $T_{n^*}\langle t_1, t_2, t_1 + t_2\rangle$ is 3-colourable in case that 3^{s+1} is not a divisor of $(t_2 - t_1)$.

For this we recall from [4] the notion of a $(K_n \setminus e)$-*cycle*.

Definition 1 Let $K_n \setminus e$ be the complete graph on n vertices with one edge removed, and let a and b denote the vertices of degree $n - 2$, which we call the *distinguished* vertices. A collection $(K^1, K^2, \ldots, K^p)$ of such $(K_n \setminus e)$s with distinguished vertices $a_1, b_1, \ldots, a_p, b_p$ is called a *$(K_n \setminus e)$-cycle*, if K^i and K^{i+1} have one of their distinguished vertices in common, i.e., $b_i = a_{i+1}$ for $i = 1, \ldots, p - 1$, and possibly $n - 3$ of its neighbors. Finally, a_1 and b_p are connected by an edge.

A $(K_n \setminus e)$-cycle C is easily seen to be *n-critical*, i.e., $\chi(C) = n$ but $\chi(C \setminus e) = n - 1$ for any edge $e \in C$. Since we only deal with Toeplitz graphs $T\langle t_1, \ldots, t_k\rangle$ with $k \le 3$, the use of $(K_4 \setminus e)$-cycles will be sufficient. The following theorem will also be useful.

Theorem 9 [4] *Let* $T = T_n\langle t_1, t_2, t_1 + t_2\rangle$ *such that* 3^{s+1} *is not a divisor of* $(t_2 - t_1)$. *Then* T *contains a* $(K_4 \setminus e)$-*cycle as a subgraph.*

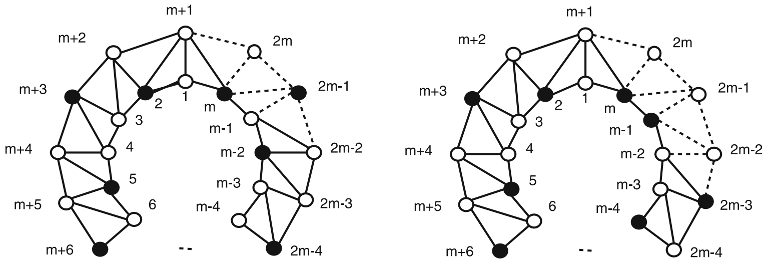


Fig. 7 The cases $r_0 = 0$ and $t_2 \cong 0 \mod 3$, $t_2 \cong 2 \mod 3$

As in the proof (in [4]) of Theorem 9, we let $t_1 + t_2 = mt_1 + r_0, 0 \leq r_0 \leq t_1 - 1$. We may also suppose that $gcd(t_1, t_2) = 1$, which reduces the hypothesis: 3^{s+1} is not a divisor of $(t_2 - t_1)$ to: 3 does not divide $(t_2 - t_1)$. Since a complete analysis seems to be very difficult, we only present a solution for $r_0 = 0$, leading to $t_1 = 1$ and the cases $t_2 \cong 0 \mod 3$ and $t_2 \cong 2 \mod 3$, that are illustrated in Fig. 7 for $n = 2(t_1 + t_2) = 2m$.

In both cases a $(K_4 \setminus e)$-cycle is easily detected as a subgraph, and a 3-colouring is impossible unless the elements $2m - 1, 2m$ in the first case, and $2m - 2, 2m - 1, 2m$ in the second are deleted. Thus, we get

Theorem 10 *The maximum number n^* for which $T_{n^*}\langle 1, t_2, t_2 + 1\rangle$ is 3-colourable, equals $2m - 2$ if $t_2 \cong 0 \mod 3$, and $2m - 3$ if $t_2 \cong 2 \mod 3$.*

For the second example of a planar Toeplitz graph T with $t_1 = 1$ as studied in Sect. 2.3 we can easily show

Lemma 11 *If $T = T_{t_3+2}\langle 1, 2, t_3, t_3 + 1\rangle$ and t_3 is even, then $2 < \chi(T) \leq 4$, and $\chi(T) = 3$ iff $t_3 \cong 1 \mod 3$.*

4 Counting Maximal Independent Sets

Given a Toeplitz graph $T = T_n\langle t_1, \ldots, t_k\rangle$, a set of vertices $I \subseteq V = \{1, \ldots, n\}$ is called an *independent set*, if $|i - j| \notin \{t_1, \ldots, t_k\}$ for all $i, j \in I$. A *maximal independent set*, or a *basis*, is an independent set with the property that $I \cup \{v\}$ is not independent any more for any $v \in V \setminus I$. Just observe that a basis of T corresponds to a maximal complete subgraph, or a *clique*, in the (edgewise) complement of T, the Toeplitz graph $\bar{T} = T\langle V \setminus \{t_1, \ldots, t_k, n\}\rangle$.

Moon and Moser [13] have shown that a graph $G = (V, E)$ with n vertices can have at most $3^{n/3}$ bases. The *exact* number $b(n)$ of bases is given in the n-vertex cycle C_n by the Perrin numbers (see Füredi [7]), and in the n-vertex path P_n by the Padovan sequence (see Euler [5]).

In the following we are going to determine $b(n)$ for several instances of Toeplitz graphs including the planar case for small values of t_k. Since the problem of counting bases in planar graphs is $\#\mathcal{P}$-complete (see Vadhan [16]), our approach may be seen as a contribution to the emerging field of fixed parameter counting.

4.1 The Case $T = T_n\langle 1, \ldots, l\rangle$

We start with a first case: $T = T_n\langle 1, \ldots, l\rangle$. T consists of a sequence of cliques of size $l+1$, and two vertices $i, j \in \{1, \ldots, n\}$ are independent, iff they are at distance $\geq l+1$. But this is precisely the way *l-independence* over the path P_n is usually defined, and the number $b(n)$ of maximal such independent sets has already been studied by Skupien (2007, Private communication).

Theorem 11 (Skupien, 2007, Private communication) *Given the Toeplitz graph $T = T_n\langle 1, \ldots, l\rangle$, the number $b(n)$ of bases satisfies the recurrence*

$$b(n) = b(n-l-1) + \cdots + b(n-2l-1) \text{ for } n \geq 2l+2,$$

with initial values

$$b(j) = j \text{ for } j = 1, \ldots, l+1,$$

$$b(l+j) = (l+1) + \binom{j-1}{2} \text{ for } j = 2, \ldots, l+1,$$

and generating function

$$\sum_{n\geq 1} b(n)x^n = \frac{\sum_{j=1}^{l+1} jx^j + x^{l+1}\sum_{k=1}^{l-1}(l+1-k)x^k}{1-\sum_{k=l+1}^{2l+1} x^k}.$$

For a proof of the recurrence, consider the path P_n over the vertex set $V = \{1, \ldots, n\}$. Partition the family $\mathcal{B}^n$ of maximal l-independent sets in P_n into $l+1$ classes $C_0, \ldots, C_l$ according to the largest element such a set does contain: this can be $n, n-1, \ldots, n-l$. Clearly, the cardinality of C_i equals $b(n-(l+1)-i)$ for $i = 0, \ldots, l$, and one easily verifies the initial conditions.

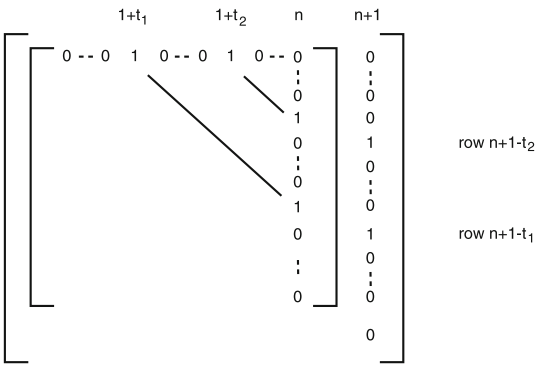
Theorem 11 thus provides recurrence formulas for the number of bases in the planar Toeplitz graphs $T_n\langle 1\rangle$, $T_n\langle 1, 2\rangle$ and $T_n\langle 1, 2, 3\rangle$.

For more general results we have adapted the transfer matrix method, well known from statistical physics (see Stanley [15] for a presentation) in a similar way as we did in Euler [5] to count the number of bases in grid graphs. The main steps can be described as follows:

1. Create a partition of $\mathcal{B}^n$ into a fixed number of classes, reproducing itself (with growing class cardinalities) when going from n to $n+1$;
2. Determine the associated transfer matrix M;
3. Calculate $det(\mathbf{1} - xM)$ to obtain a recurrence formula for $b(n)$.

To see how the different classes evolve at each step, we need to know how $\mathcal{B}^{n+1}$ arises from $\mathcal{B}^n$. As an example, consider the adjacency matrices A_n and A_{n+1} associated with $T_n\langle t_1, t_2\rangle$ and $T_{n+1}\langle t_1, t_2\rangle$ as represented in Fig. 8.

Fig. 8 Matrices A_n and A_{n+1}



Now, if

$$\mathcal{B}' = \left\{B \setminus \{n+1-t_1, n+1-t_2\} \cup \{n+1\}, B \in \mathcal{B}^n\right\},$$

then

$$\mathcal{B}^{n+1} = \left\{B \subseteq \{1, \ldots, n+1\} : B \in \mathcal{B}^n \text{ or } B \in \mathcal{B}', \text{ and } B \text{ maximal}\right\}.$$

Second, once we have partitioned $\mathcal{B}^n$ into a certain number of classes $C_0, \ldots, C_p$, when transforming $\mathcal{B}^n$ into $\mathcal{B}^{n+1}$ as indicated above, a basis $B \in C_i$, will contribute to a number of classes $C_{i_1}, \ldots, C_{i_q}$ within $\mathcal{B}^{n+1}$: if these classes are the same for every $B \in C_i$, we will call C_i a *stable* class. It is our aim to find a partition of $\mathcal{B}^n$ into stable classes. If there are p such classes, we are able to define the *transfer matrix* $M \in \{0, 1\}^{p \times p}$ as follows:

$$M_{ij} = 1 \text{ iff class } j \text{ contributes to class } i.$$

Moreover, if c_i^k denotes the cardinality of class C_i at stage k, then

$$c_i^{k+1} = \sum_{j=1}^{p} M_{ij} c_j^k \text{ for } i = 1, \ldots, p,$$

and

$$b(k+1) = \sum_{i=1}^{p} c_i^{k+1}.$$

4.2 The Case $T = T_n\langle 1, 3\rangle$

Following this approach for the Toeplitz graph $T = T_n\langle 1, 3\rangle$ we obtain a partition of $\mathcal{B}^n$ into 5 classes, as indicated in Table 1.

Table 1 Partition of $\mathcal{B}^n$, $n \geq 6$, into 5 stable classes

Classes	$B =$
C_1	$\{\sim, n-5, n\}$
C_2	$\{\sim, n-7, n-2, n\}$
C_3	$\{\sim, n-4, n-2, n\}$
C_4	$\{\sim, n-6, n-1\}$
C_5	$\{\sim, n-3, n-1\}$

The corresponding transfer matrix is

$$M = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

with $det(\mathbf{1} - xM) = 1 - x^2 - x^5$, providing a recurrence for the sequence $(b(n))_{n\in\mathbb{N}}$, together with the associated generating function.

Altogether, we obtain

Theorem 12 *Given the Toeplitz graph $T = T_n\langle 1, 3\rangle$, the number $b(n)$ of bases satisfies the recurrence*

$$b(n) = b(n-2) + b(n-5) \;\text{ for }\; n \geq 6,$$

with initial values

$i = 1, \ldots, 5$					
$b(i)$	1	2	2	2	2

defining the sequence

$$(b(n))_{n\in\mathbb{N}} = (1, 2, 2, 2, 2, 3, 4, 5, 6, 7, 9, 11, 14, 17, 21, \ldots),$$

whose generating function is

$$\sum_{n\geq 1} b(n)x^n = \frac{x + 2x^2 + x^3}{1 - x^2 - x^5}\,.$$

4.3 The Case $T = T\langle 1, 4\rangle$

The partition of $\mathcal{B}^n$ in this case is presented in Table 2.

Table 2 Partition of $\mathcal{B}^n$, $n \geq 9$, into 13 stable classes

Classes	$B =$
C_1	$\{\sim, n-8, n-5, n-2, n\}$
C_2	$\{\sim, n-7, n-5, n-2, n\}$
C_3	$\{\sim, n-9, n-2, n\}$
C_4	$\{\sim, n-7, n-2, n\}$
C_5	$\{\sim, n-6, n-3, n\}$
C_6	$\{\sim, n-5, n-3, n\}$
C_7	$\{\sim, n-9, n-6, n-3, n-1\}$
C_8	$\{\sim, n-8, n-6, n-3, n-1\}$
C_9	$\{\sim, n-10, n-3, n-1\}$
C_{10}	$\{\sim, n-8, n-3, n-1\}$
C_{11}	$\{\sim, n-7, n-4, n-1\}$
C_{12}	$\{\sim, n-6, n-4, n-1\}$
C_{13}	$\{\sim, n-4, n-2\}$

And the corresponding transfer matrix is

$$M = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0&0 \\ 1&1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1 \\ 1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&1&1&1&0&0&0 \end{bmatrix}$$

with $det(\mathbf{1} - xM) = 1 - x^3 - x^5 - x^7 - x^9 + x^{10} + x^{12}$.
Similarly to Theorem 12 we obtain

Theorem 13 *Given the Toeplitz graph $T = T_n\langle 1, 4\rangle$, the number $b(n)$ of bases satisfies the recurrence*

$$b(n) = b(n-3) + b(n-5) + b(n-7) + b(n-9) - b(n-10) \\ -b(n-12) \text{ for } n \geq 13$$

with initial values

	$i = 1, \ldots, 12$											
$b(i)$	1	2	2	3	5	5	6	7	8	11	14	18

defining the sequence

$$(b(n))_{n\in\mathbb{N}} = (1, 2, 2, 3, 5, 5, 6, 7, 8, 11, 14, 18, 23, 28, 34, 43, \ldots)\ ,$$

whose generating function is

$$\sum_{n\geq 1} b(n)x^n$$
$$= \frac{x + 2x^2 + 2x^3 + 2x^4 + 3x^5 + 2x^6 + x^7 - x^8 - 2x^9 - 3x^{10} - 2x^{11} - x^{12}}{1 - x^3 - x^5 - x^7 - x^9 + x^{10} + x^{12}}\ .$$

We observe that with increasing value of t_2 the effort to determine the sequence $(b(n))_{n\in\mathbb{N}}$ grows rapidly. It would be interesting to identify cases similar to the first one of this section, for which this effort remains reasonable.

5 Conclusion

In this paper we have described several classes of finite planar Toeplitz graphs, determined their chromatic number and given results on counting maximal independent sets for several instances of such graphs. We think that, beyond the ongoing work on colorability and hamiltonicity, future work should focus on the independence number of finite Toeplitz graphs in relation with algorithmic aspects.

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