HIGHLY NON-CONCURRENT LONGEST CYCLES IN LATTICE GRAPHS

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Abstract. There exist planar graphs in which any two vertices are missed by some longest cycle. Although this requirement is very strong, we prove here that such graphs can also be found as subgraphs of the square and hexagonal lattices. Considering (finite) such lattices on the torus and on the Möbius strip enables us to reduce the order of our examples.

Introduction

Do we have small (if possible minimal) k-connected graphs with the property that for any j vertices there is a longest path (cycle) avoiding all of them? This question was a strengthening of Gallai's question [1] and it was raised by Zamfirescu [8]. In particular, the question was (also) asked for planar graphs. After getting several answers for his questions (see [7], [3], [9]), in 2001 he asked to investigate lattice graphs from this point of view [10].

We shall say that a graph is a $\mathbf{C}_k^j$-*graph* if it is k-connected and any set of j vertices is missed by some longest cycle. The existence of $\mathbf{C}_2^1$-graphs in various lattices has been verified in [2], [4] and [5]. So far, no $\mathbf{C}_2^2$-graphs have been discovered in any lattice. It is perhaps interesting to mention that no $\mathbf{C}_2^3$-graph at all is known, so far.

In this paper, we first prove the existence of $\mathbf{C}_2^2$-graphs in the infinite square lattice $\mathcal{L}$ and hexagonal lattice $\mathcal{H}$, obviously with a considerably larger number of vertices than the smallest known $\mathbf{C}_2^2$-graph in the plane, whose order is 135, found by Zamfirescu [9].

Second, we consider (finite) square and hexagonal lattices on the torus and on the Möbius strip, which are defined according to [4], and construct $\mathbf{C}_2^2$-subgraphs of these lattices.

Of great help will be the following lemma, due to which proofs of our main results reduce to finding appropriate embeddings in the various lattices considered here.

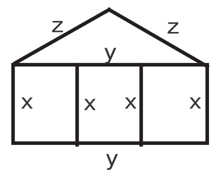

Fig. 1.

Let G be a graph homeomorphic to the graph G' in Fig. 2(b). The graph G contains ten subgraphs isomorphic to the graph of Fig. 2(a), where x, y, z, t and w are numbers of vertices of degree 2.

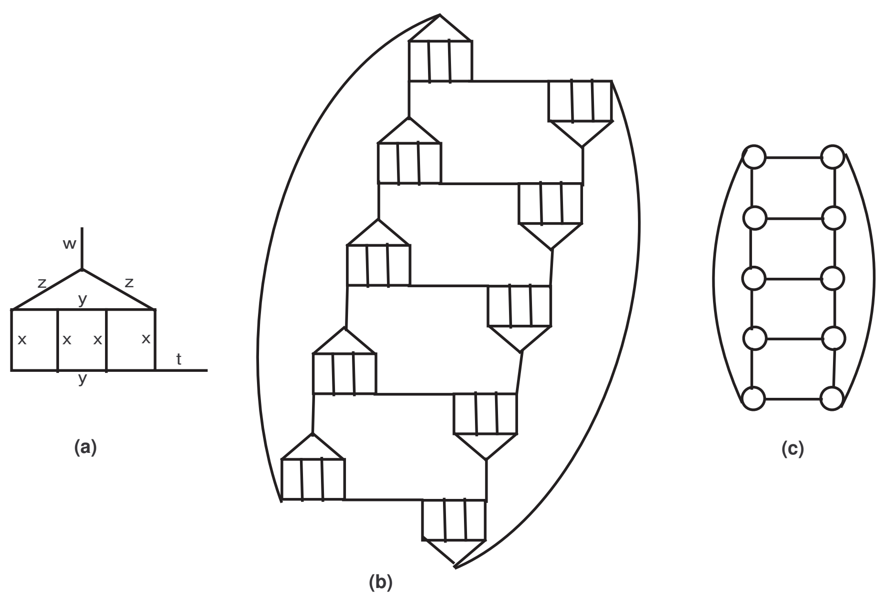

Fig. 2.

Lemma. *Any two vertices of G are missed by some longest cycle of G if $2x \geq y + 2z + 1$ and $2t = x + y + 3z + 3w + 8$.*

Proof. In the graph G, every longest cycle which passes through the subgraph, which we shall call "house", shown in Fig. 1 must adopt there one of the paths in Fig. 3.

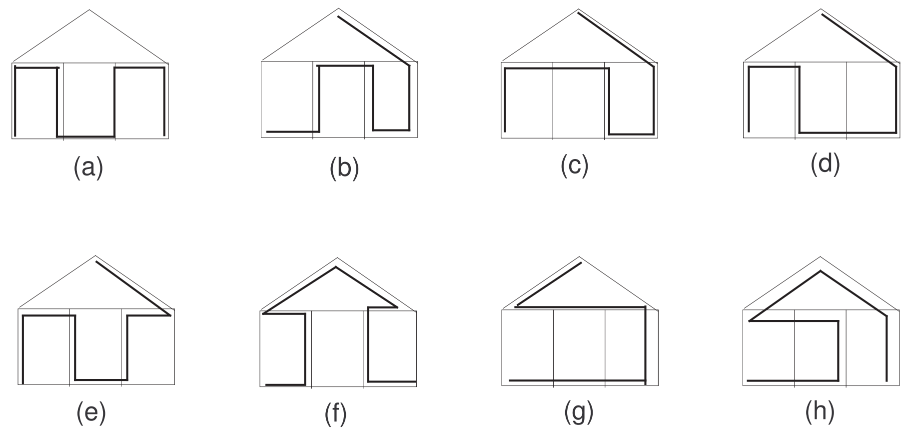

Fig. 3.

The paths from (b) to (e) of Fig. 3, are of the same length. The lengths of the paths shown in Fig. 3, (a), (b), (f), (g) and (h) are respectively

$$a = 4x + y + 7,$$

$$b = 3x + y + z + 7,$$

$$f = 2x + 2z + 8,$$

$$g = x + 2y + z + 8,$$

$$h = 2x + 2y + 2z + 8.$$

The graph G has the desired property if the following two conditions hold.

(i) Every vertex in a "house" is avoided by a path of one of the types (a), (b), (c), (d) and (e) of Fig. 3 (including paths symmetric to them).

(ii) In the graph (c) of Fig. 2, which is obtained from G' by contracting all houses, every pair of edges should be avoided by some cycle corresponding to a longest cycle of G.

Now, we compare the paths joining the same "corners" of the house.

First, we compare a, f and h. Since h is never smaller than f, and since we don't need the path of Fig. 3(h), we have the condition $a \geq h$, which means the inequality of the statement.

Second, we compare all other paths of Fig. 3, since many are equally long, what remains is $b \geq g$, because we don't need the path of Fig. 3(g). This means $2x \geq y + 1$, which follows from the previous inequality.

For any pair of edges of the graph of Fig. 2(c) there is a cycle of one of the two types shown on Fig. 4(u′), (v′) avoiding both of them. In G these become cycles like those depicted in Fig. 4(u), (v), and have lengths

$$u = 4a + 5b + 4t + 5w = 31x + 9y + 5z + 5w + 4t + 72,$$

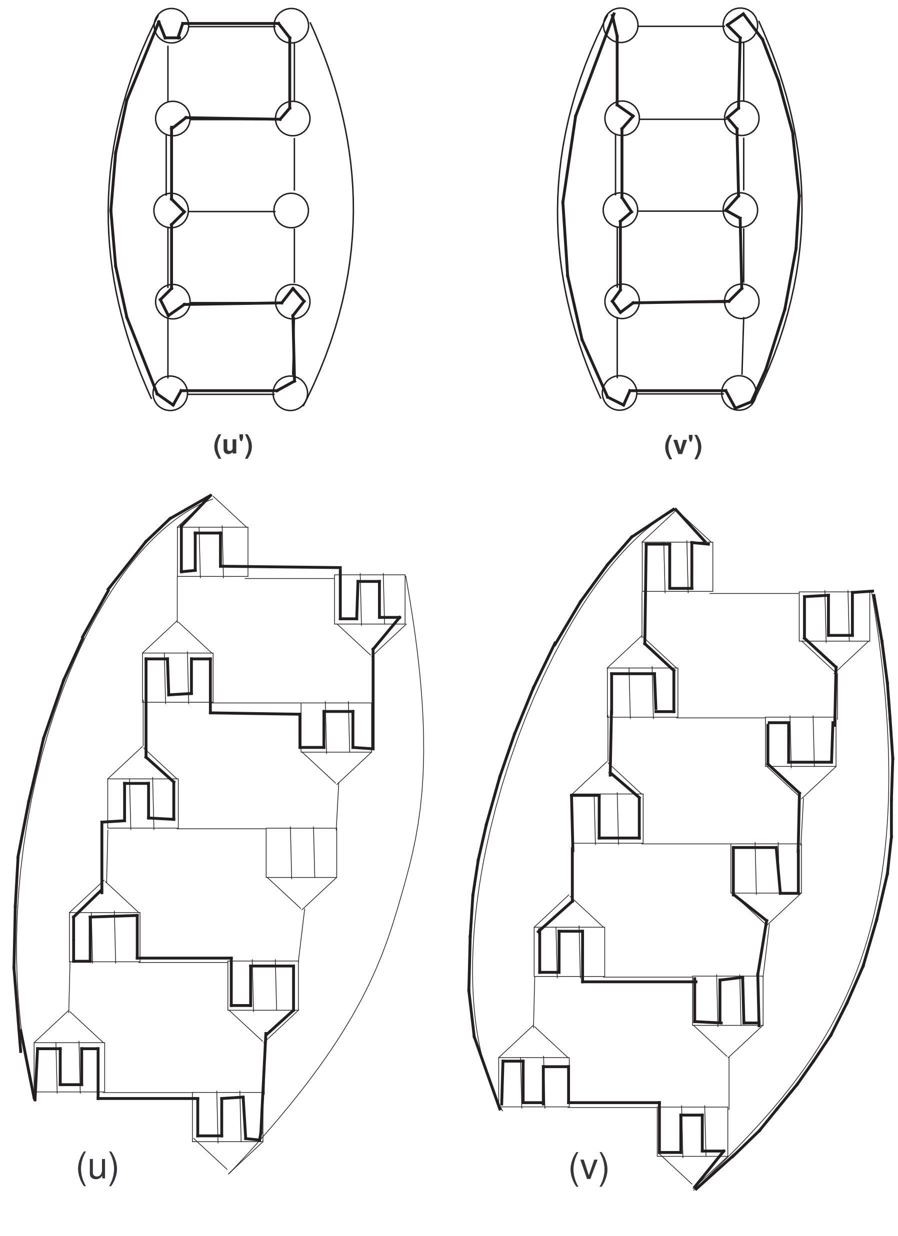

Fig. 4.

$$v = 2a + 8b + 2t + 8w = 32x + 10y + 8z + 8w + 2t + 80,$$

respectively. The equality $u = v$, is equivalent to $2t = x + y + 3z + 3w + 8$, which is the required second condition of the lemma.

Embeddings In Square Lattice Graphs

Besides the usual lattice $\mathcal{L}$ we shall also consider the toroidal lattices and Möbius strip lattices. We begin this section by the construction of toroidal and Möbius strip lattices.

To obtain the *toroidal lattice* $\mathcal{L}^T_{m,n}$, we consider an $(m+1) \times (n+1)$ rectangle (with $(m+1)(n+1)$ vertices) in $\mathcal{L}$ and identify opposite vertices on the boundary as indicated on Fig. 5(a). It has mn vertices. And to obtain a *Möbius strip lattice* $\mathcal{L}^M_{m,n}$ of order mn, we identify opposite vertices taken in reverse order in an $m \times (n+1)$ rectangle, as indicated in Fig. 5(b).

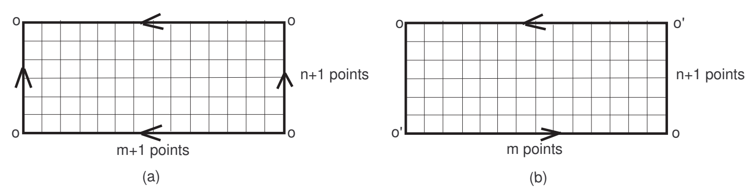

Fig. 5.

Our first result follows.

Theorem 1 *There exists a* $\mathbf{C}^2_2$*-graph of order* 490 *in* $\mathcal{L}$*.*

Proof. For $y = 1, z = 2, x = 3, t = 24$ and $w = 10$, the conditions of our Lemma are verified and the corresponding graph G is of order 490 with the desired property. Fig. 6 reveals an embedding of G in $\mathcal{L}$.

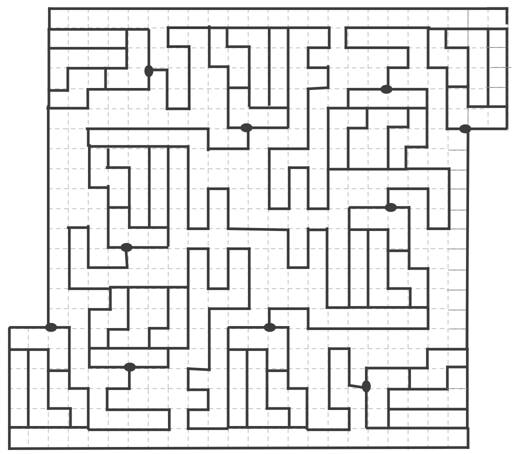

Fig. 6.

Theorem 2 *The lattice* $\mathcal{L}^T_{26,16}$ *contains a planar* $\mathbf{C}^2_2$*-subgraph of order* 315.

Proof. Let us take $y = 1, z = 2, x = 3, t = 9$ and $w = 0$ in the Lemma. It can be easily checked that the chosen values y, z, x, t and w satisfy the conditions of the Lemma and the resulting graph G is a $\mathbf{C}^2_2$-graph of order 315. Fig. 7 is an embedding of G in $\mathcal{L}^T_{26,16}$.

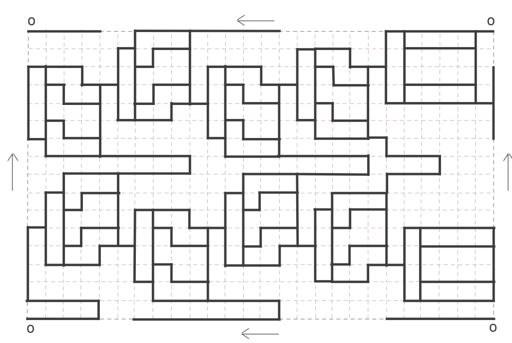

Fig. 7.

Theorem 3 *In* $\mathcal{L}^M_{34,18}$ *we have a planar* $\mathbf{C}^2_2$*-graph of order* 350.

Proof. To obtain a graph as required we will use our Lemma once again. Now we take $y = 1, z = 2, x = 3, t = 12$ and $w = 2$, which satisfy the conditions of the Lemma, and the resulting graph G is a $\mathbf{C}^2_2$-graph of order 350. Fig. 8 presents an embedding of G in $\mathcal{L}^M_{34,18}$.

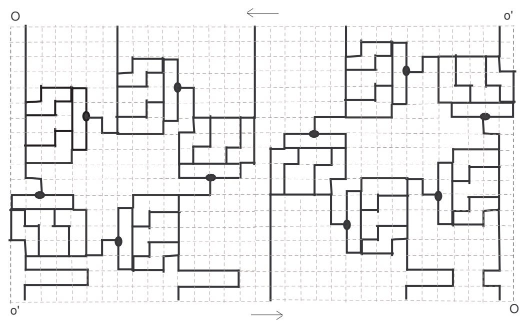

Fig. 8.

Needless to say that we made every effort to minimize the order of our graphs in all three results.

Remark 1. Notice that our graphs produced in the proofs above are all planar, even homeomorphic, but still of different orders. This is going to be equally true for the graphs obtained in the next section. The freedom of choosing non-planar graphs in the toroidal and Möbius strip lattices has not been used. Making use of this freedom may be the key for solving the Problem at the end of this paper.

Embeddings In Hexagonal Lattice Graphs

Now we are going to prove the existence of $\mathbf{C}_2^2$-graphs in $\mathcal{H}$, $\mathcal{H}_{m,n}^T$ and $\mathcal{H}_{m,n}^M$.

Just like $\mathcal{L}_{m,n}^T$ and $\mathcal{L}_{m,n}^M$, we define lattice graphs $\mathcal{H}_{m,n}^T$ and $\mathcal{H}_{m,n}^M$ of order mn according to Fig. 9(a) and Fig. 9(b) respectively.

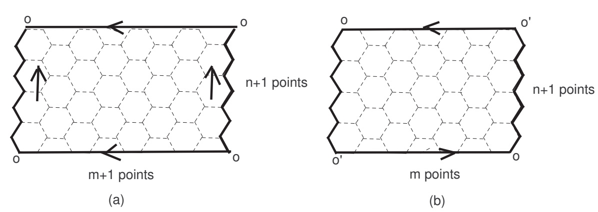

Fig. 9.

Theorem 4 *The lattice $\mathcal{H}$ contains a $\mathbf{C}_2^2$-subgraph of order* 950.

Proof. Once again we consider our Lemma and take $y = z = 3, x = 8, t = 44$ and $w = 20$. This time the resulting graph G is a $\mathbf{C}_2^2$-graph of order 950. An embedding of G in $\mathcal{H}$ is shown in Fig. 10.

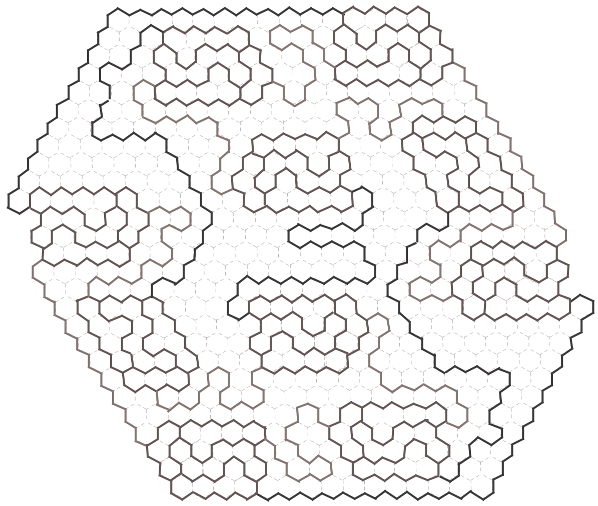

Fig. 10.

Theorem 5 *There exists a planar* $\mathbf{C}_2^2$*-subgraph of* $\mathcal{H}_{30,32}^T$ *of order* 600*.*

Proof. We use again a particular case of the Lemma as its conditions are also satisfied for $y = z = 3, x = 8, t = 14$ and $w = 0$, and the corresponding graph G is of order 600. Fig. 11 shows an embedding of G in $\mathcal{H}_{30,32}^T$.

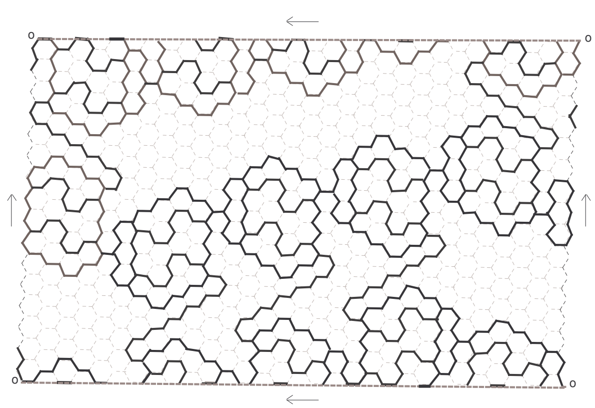

Fig. 11.

Theorem 6 *In* $\mathcal{H}_{42,28}^M$ *we have a planar* $\mathbf{C}_2^2$*-graph of order* 670*.*

Proof. The conditions of the Lemma are also verified if we take $y = z = 3, x = 8, t = 20$ and $w = 4$, and we are led to a graph G of order 670. Fig. 12 reveals an embedding of G in $\mathcal{H}_{42,28}^M$.

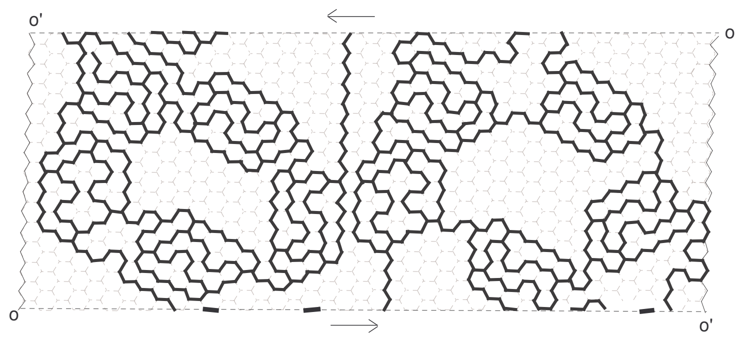

Fig. 12.

Remark 2. Fig. 13 and Fig. 14 prove that the lattices $\mathcal{L}$ and $\mathcal{H}$ contain a $\mathbf{C}_1^j$-subgraph for each $j \geq 1$ of order $4j+4$ and $6j+6$ respectively. These subgraphs are homeomorphic to Thomassen's examples in [6].

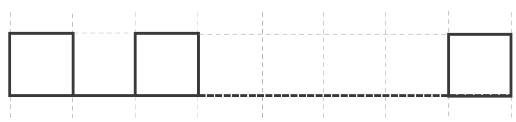

Fig. 13. A connected graph consists of (j+1) vertex disjoint C_4 cycles.

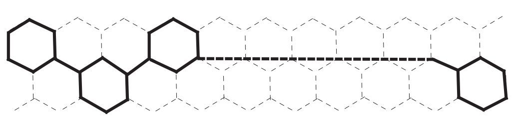

Fig. 14. A connected graph consists of (j+1) vertex disjoint C_6 cycles.

We conclude this paper with the following problem.

Problem. Find embeddings of smaller order than those presented in Theorems $1-6$. Moreover, find lattices of smaller order than those of Theorems $2, 3, 5, 6,$ admitting the desired embeddings.

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