Every point is critical

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Abstract. We show that, for any compact Alexandrov surface S (without boundary) and any point y in S, there exists a point x in S for which y is a critical point. Moreover, we prove that uniqueness characterizes the surfaces homeomorphic to the sphere among smooth orientable surfaces.

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Introduction. In this paper, by *surface* we always mean a compact 2-dimensional Alexandrov space with curvature bounded below and without boundary, as defined by Burago, Gromov and Perelman in [1]. It is known that our surfaces are topological manifolds (see [1], §11). Let $\mathcal{A}$ be the space of all surfaces.

For any surface S, denote by ρ its metric, and by ρ_x the distance function from x, given by $\rho_x(y) = \rho(x, y)$. A point $y \in S$ is called *critical* with respect to ρ_x (or to x), if for any direction τ of S at y there exists a *segment* (i.e., a shortest path) from y to x whose direction at y makes an angle not greater than $\pi/2$ with τ. For the definition of the set of directions at an arbitrary point of an Alexandrov surface, see again [1]. A *geodesic* is a curve which is locally a segment. We recall that geodesics of S do not bifurcate [1].

The survey [2] by K. Grove presents the principles, as well as applications, of the critical point theory for distance functions.

Every point on a surface admits a critical point. It suffices, indeed, to take a point farthest from it. Conversely, is it true that every point is a critical point of some other point? Certainly, not every point on every surface is a farthest point from some other point: On an ellipsoid of revolution with an NS-axis much longer than the other two, no point of the equator is farthest from any other point. Concerning the set of all critical points, however, the answer is affirmative, as Theorem 1 shows.

For the set-valued function associating to each point of a surface the set of all farthest points on the surface, the relationship between being single-valued and being surjective is investigated in [11].

Theorem 2 characterizes the smooth orientable surfaces homeomorphic to the sphere.

For any point x in S, denote by Q_x the set of all critical points with respect to x, and by Q_x^{-1} the set of all points $y \in S$ with $x \in Q_y$. Let M_x, F_x be the sets of all relative, respectively absolute, maxima of ρ_x. For properties of Q_x and its subsets M_x and F_x in Alexandrov spaces, see [3], [11], and the survey [9].

A forthcoming paper, [4], will provide for orientable surfaces an upper bound for cardQ_y^{-1} depending on the genus, and use it to estimate the cardinality of diametrally opposite sets on S. The case of points y in orientable Alexandrov surfaces, which are common maxima of several distance functions, is treated in [10].

We denote by T_y the space of directions at $y \in S$; the length λT_y of T_y satisfies $\lambda T_y \leq 2\pi$ [1]. If $\lambda T_y < 2\pi$ then y is called a *conical point* of S. A surface without conical points is called *smooth.*

There might exist a direction $\tau \in T_y$ such that no segment starts at y in direction τ. On most convex surfaces, the set of such directions τ, called *singular*, is even residual in T_y, for each y (see Theorem 2 in [12]). However, the set of non-singular directions is always dense in T_y. For those τ, for which there is a geodesic Γ with direction τ at y, a so-called *cut point* $c(\tau)$ is associated, defined by the requirement that the arc $yc(\tau) \subset \Gamma$ is a segment which cannot be extended further beyond $c(\tau)$ (remaining a segment). This is well-defined, because in an Alexandrov space of curvature bounded below segments (and geodesics) do not bifurcate. The set of cut points in all non-singular directions at y is the *cut locus* $C(y)$ of the point y.

Recall that a *tree* in S is a set $T \subset S$ any two points of which can be joined by a unique Jordan arc included in T. A set $L \subset S$ is a *local tree* if each of its points x has a neighbourhood V in S such that the connected component $K_x(V)$ of $L \cap V$ containing x is a tree. The *degree* of a point x of a local tree is the cardinality of the set of components of $K_x(V) \setminus \{x\}$ if the neighbourhood V of x is chosen such that $K_x(V)$ be a tree. A point of the local tree L is called an *extremity* of L if it has degree 1, and a *ramification point* of L if it has degree at least 3.

It is known that $C(y)$, if it is not a single point, is a local tree (see [8], Theorem A, p. 534), even a tree if S is homeomorphic to the sphere (which is easily seen). Theorem 4 in [14] and Theorem 1 in [12] yield the existence of surfaces S on which the set of all extremities of any cut locus is residual in S. It is known, however, for any surface S and point $y \in S$, that $C(y)$ has an at most countable set $C_3(y)$ of ramification points (see [8], Theorem A, p. 534).

If S is not a topological sphere, the *cyclic part of* $C(y)$ is the minimal (with respect to inclusion) subset $C^{cp}(y)$ of $C(y)$, whose removal from S produces a topological (open) disk. It was introduced by some of us in [5]. Let $C_3^{cp}(y)$ be the set of points of degree at least 3 in $C^{cp}(y)$.

It is possible that, for some point $x \in C(y)$, there exists a whole nondegenerate arc $A \subset T_y$ such that $c(\tau) = x$ for all $\tau \in A$, providing a *pencil of segments* from y to x. Let $C^*(y)$ denote the set of all such points in $C(y)$. Also, let $C^e(y)$ be the set of extremities of $C(y)$.

The points in $Cl(y) = C(y) \setminus (C_3(y) \cup C^*(y) \cup C^e(y))$ are called *cleave points.*

Main result. In order to prove the main result of the paper we need the following lemmas.

Lemma 1. *Let y belong to a surface S not homeomorphic to the sphere. Then $C^{cp}(y)$ is a local tree with no extremities and with finitely many ramification points, each having finite degree in $C^{cp}(y)$.*

This essentially follows from Myers' early investigation of the topological properties of the cut locus [7], and the observation that $C(y) \setminus C^{cp}(y)$ is a (countable) union of disjoint trees.

Lemma 2. *If $y \in S$, then all points of $C(y)$ but at most countably many are cleave points or extremities. Moreover, for any cleave point x, there are exactly two distinct directions $\tau_1(x), \tau_2(x) \in T_y$ with $c(\tau_1(x)) = c(\tau_2(x)) = x$.*

The first part follows from the already mentioned at most countability of $C_3(y)$ and from the obvious countability of $C^*(y)$. The second part is straightforward.

Lemma 3. *Let $S \in \mathcal{A}$, $y \in S$, and $x \in C(y)$. Assume that there are two segments σ_1, σ_2 from y to x such that $S \setminus (\sigma_1 \cup \sigma_2)$ be disconnected, at least one of its components being a topological open disc Δ. If the angle between σ_1 and σ_2 at y toward Δ is larger than π, then $y \in Q_x$ or there exists a point in $C(y) \cap \Delta$ with respect to which y is critical.*

Proof. We will identify in this paper T_y with a circle of centre $\mathbf{0}$ and length $\lambda T_y \leq 2\pi$ in $\mathbf{R}^2$. Let $D \subset \mathbf{R}^2$ be the compact disc bounded by T_y.

We want to find $x \in C(y)$, for which $y \in Q_x$. This is equivalent to $\mathbf{0} \in \mathrm{conv} c^{-1}(x)$, by the definition of a critical point.

Let $A^* \subset T_y$ be the arc of all directions toward the closure of Δ.

If $c(\tau) = x$ for all $\tau \in A^*$, then $y \in Q_x$. In the contrary case, if all components of $A^* \setminus c^{-1}(x)$ have length at most π, then again $y \in Q_x$. Suppose one component, A, has length $\lambda A > \pi$. For any non-singular $\tau \in A$, $c^{-1}(c(\tau)) \subset A$, because no segment starting at y in a direction belonging to A meets again any segment starting at y in a direction not belonging to A (since segments don't bifurcate). This follows from the Jordan curve theorem and the fact that Δ is homeomorphic to an open disc. Let B_τ be the shortest subarc of A including $c^{-1}(c(\tau))$ (possibly reduced to $\{\tau\}$).

Choose a non-singular direction τ_0 in the interior of A at distance (measured on T_y) less than $(\lambda T_y - \lambda A)/2$ from the mid-point of A. Then either

i) $\mathbf{0} \in \mathrm{conv} c^{-1}(c(\tau_0))$, or

ii) $B_{\tau_0} = \{\tau_0\}$ (and $c(\tau_0)$ is an extremity of $C(y)$), or else

iii) $0 < \lambda B_{\tau_0} < \lambda T_y/2$.

Put $x' = c(\tau_0)$.

In the first case *i)*, $y \in Q_{x'}$. In the last two cases, there is a single Jordan arc $J \subset C(y)$ from x to x'. The multivalued mapping $z \mapsto c^{-1}(z)$ defined on J is upper semicontinuous. For this reason, if $z \in J \setminus C_3(y)$ is close to x and $\tau \in c^{-1}(z)$, then $\lambda B_\tau > \pi \geq \lambda T_y/2$. For the same reason, if $z \in J \setminus C_3(y)$ is close to x' and $\tau \in c^{-1}(z)$, then $\lambda B_\tau < \lambda T_y/2$. Once again due to the upper semicontinuity of the above mapping, we have the intermediate value theorem in the form that, in between, there exists a point $z_0 \in J$ for which $\mathbf{0} \in \mathrm{conv} c^{-1}(z_0)$.

Theorem 1. *Every point on every surface is critical with respect to some point of the surface.*

Proof. Let $S \in \mathcal{A}$ and $y \in S$. We keep the notation from the proof of Lemma 3.

Case 1. S is homeomorphic to the sphere.

If $C(y)$ is a single point, the conclusion is true. Suppose $C(y)$ is not a point, but remember it is a tree.

Choose a point $x \in C(y)$ different from an extremity of $C(y)$. If all components of $T_y \setminus c^{-1}(x)$ have length at most π, then $y \in Q_x$. If one component has length larger than π, then the conclusion follows from Lemma 3.

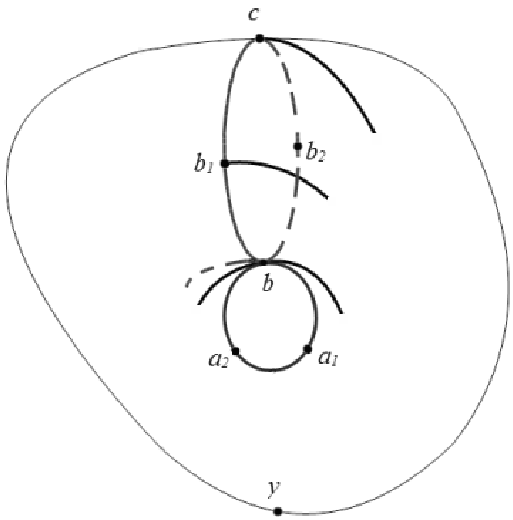

Fig. 1: Cut locus $C(y)$ of a point y on a torus. Here, $C_3(y) = \{b, b_1, c\}$, and $C_3^{cp}(y) = \{b\}$. The cyclic part $C^{cp}(y)$ consists of two cycles. The degree of b in $C(y)$ is 7, and in $C^{cp}(y)$ is 4.

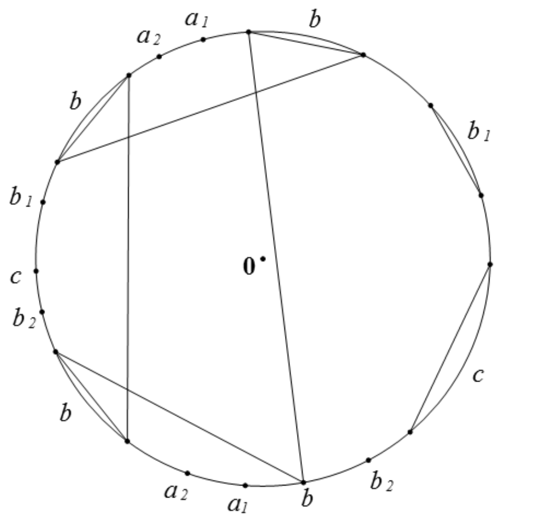

Fig. 2: This is T_y corresponding to Fig. 1. There are four distinct arcs $I(\alpha)$ with $c(\alpha) = b$, denoted here simply by b, three of which are non-degenerate. There are two distinct arcs $I(\alpha)$ with $c(\alpha) = b_1$, denoted here by b_1, one degenerate and one not.

Case 2. S is homeomorphic to the projective plane.

In this case $C^{cp}(y)$ is a closed Jordan curve on S. If $x \in C^{cp}(y)$ is a ramification point of $C(y)$, then at least three segments join x to y, at least two of which come locally from the same side of $C^{cp}(y)$ and thus bound a topological disc.

For any $x \in C^{cp}(y)$ and $\alpha \in c^{-1}(x)$, let $I(\alpha) = \alpha_-\alpha_+ \subset T_y$ be the maximal arc containing α such that, for each non-singular $\tau \in I(\alpha)$, either $c(\tau) \notin C^{cp}(y)$ or $c(\tau) = x$. (The indices $-, +$ are taken according to a certain orientation of T_y. We will say that α_- is the left endpoint of the arc $I(\alpha)$, while α_+ is its right endpoint.) In other words, $I(\alpha)$ is the maximal arc $A' \subset T_y$ containing α, so that $c(A'') \cap C^{cp}(y)$ be a single point, where A'' is the set of all non-singular directions in A'. Of course, $I(\alpha)$ may well be reduced to the singleton $\{\alpha\}$. Then $\alpha_- = \alpha_+ = \alpha$. This certainly happens if x is a cleave point, hence at all but at most countably many points of $C^{cp}(y)$. If $\alpha_- \neq \alpha_+$, then there is a pencil of segments from x to y, or x is a ramification point of $C(y)$, or both.

If the arc $I(\alpha)$ has length more than π, either $x \in Q_y^{-1}$ in the case of the pencil, or we find a point in Q_y^{-1} on a branch (i.e. component of $C(y) \setminus C^{cp}(y)$) of $C(y)$ lying between the segments from y to x in directions α_- and α_+, by Lemma 3.

Assume now that $\lambda I(\alpha) \leq \pi$ for all $\alpha \in c^{-1}(x)$ and $x \in C^{cp}(y)$.

For any $x \in C^{cp}(y) \cap Cl(y)$, there are precisely two directions $\tau_1(x)$ and $\tau_2(x)$ with $c(\tau_1) = c(\tau_2) = x$, by Lemma 2. When x describes $C^{cp}(y) \cap Cl(y)$, moving from a position x_0 along $C^{cp}(y)$ until it reaches again x_0, then $\tau_1(x)$ moves from $\tau_1(x_0)$ to $\tau_2(x_0)$, jumping over arcs like $I(\alpha)$, and $\tau_2(x)$ moves from $\tau_2(x_0)$ to $\tau_1(x_0)$, also jumping over similar arcs.

If $\mathbf{0}$ was, say, on the right hand side of the line-segment $\overline{\tau_1(x)\tau_2(x)}$ when x started its motion, $\mathbf{0}$ ends up being on its left side. Therefore, for some position x, either $\mathbf{0}$ is on $\overline{\tau_1(x)\tau_2(x)}$ or, more generally, $\mathbf{0} \in \text{conv}(I(\alpha) \cup I(\beta))$ for some subarcs $I(\alpha)$, $I(\beta)$ of T_y with $c(\alpha) = c(\beta) = x$. In both cases $y \in Q_x$.

Case 3. S is neither homeomorphic to the sphere, nor to the projective plane.

In the proof of this case, to simplify notation, we shall write C, C^{cp}, Cl, C_3^{cp} instead of $C(y)$, $C^{cp}(y)$, $Cl(y)$, $C_3^{cp}(y)$, respectively, as the point y remains the same all the way. See Figures 1, 2.

Take a point $x \in C^{cp} \setminus Cl$, and a direction $\alpha \in c^{-1}(x)$. (We have $C^{cp} \neq \emptyset$, as we are not in Case 1.) Notice that $C^{cp} \setminus Cl \supset C_3^{cp}$.

Consider first the case that $x \notin C_3^{cp}$. We recall that $I(\alpha) = \alpha_-\alpha_+ \subset T_y$ is the maximal arc containing α such that, for each non-singular $\tau \in I(\alpha)$, either $c(\tau) \notin C^{cp}$ or $c(\tau) = x$. If $I(\alpha)$ is not reduced to the singleton $\{\alpha\}$, we connect α_- to α_+ by a line segment $\overline{\alpha_-\alpha_+}$ in D. (There must be some $\alpha \in c^{-1}(x)$ with $\alpha_- \neq \alpha_+$, since x is not a cleave point.)

Consider now the case $x \in C_3^{cp}$. (If $C_3^{cp} = \emptyset$ then C^{cp} is a cycle and S is homeomorphic to the projective plane, which we now exclude.)

For each such x we have finitely many arcs like $I(\alpha)$, perhaps degenerate, as many as the degree of x in C^{cp}. For each arc $I(\alpha)$, we connect α_- (resp. α_+) to one endpoint of another arc $I(\beta)$ with $\beta \in c^{-1}(x)$, $\beta \neq \alpha$ (resp. $I(\gamma)$ with $\gamma \in c^{-1}(x)$, $\gamma \neq \alpha$) in the following way.

By symmetry, it suffices to explain this for α_-. Let $z \in C^{cp} \cap Cl$ converge to x in such a way that $\tau_1(z)$ converges to α_- from the left. Then $\tau_2(z)$ tends to some point $f(\alpha_-)$ in $c^{-1}(x)$. It is not hard to see that this point $f(\alpha_-)$ is an endpoint of another arc $I(\beta)$, moreover $\tau_2(z)$ tends to β_- from the left if the endpoint $f(\alpha_-)$ is β_-, and to β_+ from the right if $f(\alpha_-) = \beta_+$. We define this function f for α_+ and for all other endpoints of arcs like $I(\alpha)$, with $c(\alpha) = x$ (while x remains fixed).

Connect α_- to $f(\alpha_-)$ by a line-segment $\overline{\alpha_- f(\alpha_-)}$ and α_+ to $f(\alpha_+)$ by a line-segment $\overline{\alpha_+ f(\alpha_+)}$.

When the same process is applied to the endpoint $f(\alpha_-)$ of $I(\beta)$, we obtain $f(f(\alpha_-)) = \alpha_-$, so its connection with the endpoint α_- of $I(\alpha)$ is confirmed. This is due to the fact that for $z \in C^{cp} \cap Cl$ the map $\tau_1(z) \mapsto \tau_2(z)$ is an involution.

This shows that the union of the line-segments $\overline{\alpha_-\alpha_+}, \overline{\alpha_- f(\alpha_-)}, \overline{\alpha_+ f(\alpha_+)}$ and their analogs, for fixed $x \in C_3^{cp}$, forms a cycle (closed polygonal line in D). Now we do this for all (finitely many) $x \in C_3^{cp}$. So, we obtain a finite set of cycles. Their vertices on T_y are endpoints of (perhaps degenerate) arcs like $I(\alpha)$.

Let now $I(\alpha), I(\beta)$ be two arcs on T_y with $c(\alpha) = x_1, c(\beta) = x_2$ and $x_1, x_2 \in C_3^{cp}$, (where x_1 and x_2 are different or not) so that α_+, β_- are consecutive on T_y. Then, as it is easy to check, $f(\alpha_+)$ and $f(\beta_-)$ are also consecutive on T_y. Consider the cycle made up by the arcs $\alpha_+\beta_-$ and $f(\alpha_+)f(\beta_-)$, and the line-segments $\overline{\beta_- f(\beta_-)}$, $\overline{\alpha_+ f(\alpha_+)}$, and all analogous cycles, in addition to the previous ones.

Moreover, consider the cycle formed by the arc $\alpha_-\alpha_+$ and the line-segment $\overline{\alpha_-\alpha_+}$, plus all analogous cycles, not only for $c(\alpha) \notin C_3^{cp}$, but also for $c(\alpha) \in C_3^{cp}$.

Let $C_1, ..., C_n$ be all the cycles defined above. Each of them is either a closed polygonal line, or a 2-cycle formed by one subarc of T_y and one line-segment, or a 4-cycle formed by two subarcs of T_y and two line-segments.

When τ runs along T_y, $c(\tau)$ describes various parts of C.

When τ runs along an arc $I(\alpha)$, $c(\tau)$ either is constant (in the case of a pencil of segments), or runs through a tree in C, starting and ending at the ramification point $c(\alpha)$ of C. The presence of a cycle which is a closed polygonal line indicates that its vertices are directions of segments all ending at the same ramification point of C^{cp}.

If $\mathbf{0} \in \cup_{j=1}^n C_j$, i.e. $\mathbf{0}$ lies not inside but on some cycle, then $\mathbf{0}$ belongs to one of the line-segments, as $\mathbf{0} \notin T_y$. Hence $c^{-1}(x)$ contains, for some $x \in C^{cp} \setminus Cl$, two diametrally opposite points of T_y, and we are done.

If not, consider the odd-even winding number, or degree modulo 2 (as it is called in Milnor's book [6], page 20), $w(C_j) = w(\mathbf{0}, C_j)$ of the cycles C_j with respect to $\mathbf{0}$. We have

$$\sum_{i=1}^{n} w\left(C_i\right) = w\left(\sum_{i=1}^{n} C_i\right) = w\left(T_y\right) = 1,$$

because each line-segment is used exactly twice and each arc in T_y exactly once. This shows that $w(C_i) = 1$ for some cycle C_i.

Assume this cycle C_i is of the form $\alpha_+\beta_- f(\beta_-) f(\alpha_+)$, where $c(\alpha) = x_1$ and $c(\beta) = x_2$. We let z move on the arc connecting x_1 to x_2 in C^{cp}, avoiding non-cleave points, so $c^{-1}(z) = \{\tau_1(z), \tau_2(z)\}$ is well-defined. Remember that c^{-1} is upper semi-continuous everywhere.

If $\alpha_+\beta_-$ and $f(\beta_-)f(\alpha_+)$ are of the same orientation on T_y, then the proof parallels that of Case 1: $\tau_1(z) \in \alpha_+\beta_-$ and $\tau_2(z) \in f(\alpha_+)f(\beta_-)$ move in contrary directions.

If $\alpha_+\beta_-$ and $f(\beta_-)f(\alpha_+)$ are of opposite orientations on T_y, then $\tau_1(z)$ and $\tau_2(z)$ move in the same direction, but $\mathbf{0}$ lies on different sides of the line-segment $\overline{\tau_1(z)\tau_2(z)}$ when z is close to x_1 and when it is close to x_2; this and the argument of Case 2 yield the conclusion.

Assume next that C_i is a cycle $\alpha_-\alpha_+ \cup \overline{\alpha_+\alpha_-}$. Then the conclusion follows from Lemma 3.

Finally, if C_i is one of the other cycles (with all edges line-segments), then $w(C_i) = 1$ means that $\mathbf{0}$ is surrounded by C_i, which is impossible if $\mathbf{0} \notin \mathrm{conv}C_i$. By construction, $\mathrm{conv}C_i = \mathrm{conv}c^{-1}(x)$ for some $x \in C_3^{cp}$.

The proof is complete.

A characterization of the sphere. The following lemma shows that in general one cannot hope for a better lower bound. It extends Theorem 3 in [11] and admits a similar proof, which will therefore be omitted.

Lemma 4. *Assume $S \in \mathcal{A}$, $y \in S$ is not conical, and $x \in Q_y^{-1}$ is such that the union U of two segments from x to y separates S. If a component S' of $S \setminus U$ contains no segment from x to y then $Q_y^{-1} \cap S' = \emptyset$. In particular, if the union of any two segments from x to y separates S then $Q_y^{-1} = \{x\}$.*

Theorem 2. *A smooth orientable surface S is homeomorphic to the sphere if and only if each point in S is critical with respect to precisely one other point of S.*

Proof. If S is homeomorphic to the sphere, then $\mathrm{card}Q_y^{-1} \geq 1$ by Theorem 1. Using now Lemma 4, we obtain $\mathrm{card}Q_y^{-1} = 1$.

Next, we show that every orientable surface of positive genus contains a point y with $\mathrm{card}Q_y^{-1} > 1$.

To see this, let Ω denote a shortest simple closed curve which does not separate S. Then Ω is a closed geodesic. Moreover, for any of its points z, Ω is the union of two segments of length $\lambda\Omega/2$ starting at z and ending at z_Ω. Consider the family $\mathcal{C}$ of all simple closed not contractible curves C which cut Ω at precisely one point, such that Ω separates C locally at $\Omega \cap C$. Then clearly $\mathcal{C} \neq \emptyset$, by the choice of Ω. Let Ω' be a shortest curve in $\mathcal{C}$; it is a closed geodesic too. Moreover, by the definition of $\mathcal{C}$ and by the choice of Ω', the latter is the union of two segments starting at $\{y\} = \Omega \cap \Omega'$ and ending at $y_{\Omega'}$. It follows that Q_y^{-1} contains at least two points, y_Ω and $y_{\Omega'}$.

Open problem. Every smooth orientable surface of positive genus possesses points x, y such that y is critical with respect to x and two segments from y to x have opposite directions at y (see the proof of Theorem 2). Is the same true for all smooth surfaces homeomorphic to the sphere? Or, at least, if $\mathcal{A}_0$ denotes the space of all Alexandrov surfaces homeomorphic to the sphere, endowed with the Hausdorff-Gromov metric, is there a dense set in $\mathcal{A}_0$ with the above property? For convex surfaces, this problem was raised in [13], and is still open.

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