Balanced triangulations

Lijie JIA, Liping YUAN, Carol T. ZAMFIRESCU,
and Tudor I. ZAMFIRESCU

Abstract. Motivated by applications in numerical analysis, we investigate balanced triangulations, i.e. triangulations where all angles are strictly larger than $\pi/6$ and strictly smaller than $\pi/2$, giving the optimal lower bound for the number of triangles in the case of the square. We also investigate platonic surfaces, where we find for each one its respective optimal bound. In particular, we settle (affirmatively) the open question whether there exist acute triangulations of the regular dodecahedral surface with 12 acute triangles [Itoh and Zamfirescu, *Europ. J. Combin.* **28** (2007)].

Key Words. Balanced triangulations, acute triangulations, platonic solids.

MSC 2010. 52C20, 52C99.

Introduction

A *triangulation* of a 2-dimensional space means a collection of (full) triangles covering the space such that the intersection of any two triangles is either empty, or a vertex or an edge. We call a triangulation *geodesic*, if all its triangles are geodesic, meaning that their edges are *segments*, i.e. shortest paths between the corresponding vertices. In this paper we shall always refer to geodesic triangulations. A triangulation is called *acute* if all of its angles are acute. We have a *balanced* triangulation, if all angles of the geodesic triangles are strictly larger than $\pi/6$ and strictly smaller than $\pi/2$. The lower bound of $\pi/6$ for the angles is especially appealing, because together with the upper bound of $\pi/2$ it bounds the relevant ratio between the shortest side and the longest side of every triangle, from below, by 1:2. This brings us into the frame of "bounded geometry", which deals with objects having a bounded ratio of edge-lengths (see, for example, [1], [12], and [6]). A recent survey on acute and non-obtuse triangulations (the latter having angles not larger than $\pi/2$), is [14] (see also [15]).

The motivation for selecting bounds both from above and from below stems from mesh generation applications, anchored in numerical analysis. Very flat and very sharp angles are undesirable.

We shall from now on use in this paper angle measures in degrees.

Concerning algorithmic approaches on the non-obtuse triangulation of polygons with n sides, Baker, Grosse, and Rafferty [2] presented in 1988 an algorithm yielding a triangulation with angles no smaller than 13° and no larger than 90°, as long as the smallest angle of the input polygon had at least 13°. Bern, Mitchell, and Ruppert proved in [3] that there exists an algorithm creating a triangulation which requires only $O(n)$ triangles, improving on previous results. Yuan found in [13] a concrete upper bound for the size of a non-obtuse triangulation of an n-gon based on work in [3], namely $106n - 216$. She also proved that one can transform any non-obtuse triangulation needing N triangles into an acute one requiring $22N$ triangles.

The vertices and the edges of a triangulation form a graph. For a vertex v, $d(v)$ denotes its degree.

In the plane, by $a_1a_2...a_n$ we denote the (full) n-gon with vertices a_1, a_2,..., a_n, i.e. the convex hull of $\{a_1, a_2, ..., a_n\}$.

For points p_1, p_2 we denote by $|p_1p_2|$ the distance from p_1 to p_2.

The square

We begin with the square. We know from [4] that a square needs at least eight triangles to be acutely triangulated. Cassidy and Lord [4] also proved that there exists no acute triangulation with nine triangles, but that there exists an acute triangulation with n triangles for every n larger than 9.

We mention here that Eppstein discusses a slightly different problem (see [5]), posed initially by Tromp in 1996: how to make the angles as acute as possible (i.e. minimizing the maximal angle). For the eight-triangle solution, he found positions of the vertices for which the maximum angle is only about 85°, and asked if more triangles would achieve even better angles. Motivated by Tromp's question and a result by Gerver [7] (who shows how to find a dissection – not a triangulation! – of a polygon with no angles larger than 72°, assuming all angles of the input measure at least 36°), Eppstein produces a triangulation requiring 14 triangles, with all angles measuring 45°, 54°, 63°, or 72°.

Theorem 1. *The square admits a balanced triangulation of size* 11*, and this is best possible.*

Proof. We first show that there exists a balanced triangulation of the square with 11 triangles. Consider the triangulated square depicted in Fig. 1, and apply gentle shifts in the directions of the arrows, in the indicated order. It is now easy to see that for any angle α of the triangulation we have $44° < \alpha < 90°$.

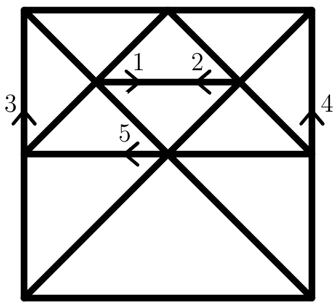

Fig. 1: A balanced triangulation of the square

The second part of this proof deals with the minimality of the size. In view of Cassidy and Lord's result [4], it would suffice to show that there exists no balanced triangulation of the square with eight or ten triangles, but our proof will be independent of their result.

Let Q be the given square, and a, b, c, e its vertices in sinistrorsum order.

Notice that $d(a) = d(b) = d(c) = d(e) = 3$, while the degree of the other boundary vertices is at least 4, and the degree of vertices interior to Q is at least 5.

Remark ($\diamondsuit$): No edge uv traverses int$Q$ joining two boundary vertices which lie on consecutive sides of Q, because, if a, say, is the common vertex of the two consecutive sides, and wuv is the triangle included in auv having uv as side, then $\angle uwv > 90°$, a contradiction.

Suppose just one vertex v of the triangulation lies in int$Q$. As $d(v) \geq 5$, there must exist a vertex $w \in$ bd$Q$ different from a, b, c, e, say $w \in ab$. As $d(w) \geq 4$, there is a vertex $u \neq w$ joined to w.

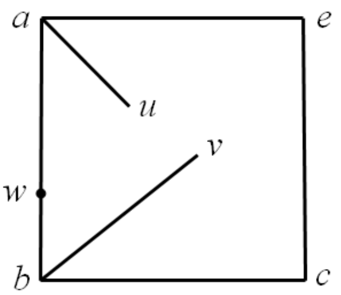

Fig. 2

By Remark ($\diamondsuit$), $u \in ec \setminus \{e, c\}$. Assume w.l.o.g. that $v \in waeu$. Then, in $wbcu$, there are extra edges starting at b, c, and either w or u. This is impossible in the absence of vertices interior to $wbcu$.

Now suppose just two vertices u, v lie in int$Q$. If there is no vertex in bd$Q$ distinct from a, b, c, e, then both u, v are joined to a, b, c, e, which is impossible.

Let $w \in$ bd$Q$ be a vertex, say $w \in ab$. By Remark ($\diamondsuit$), there is a single vertex u in int$Q$ adjacent to a. If ub is also an edge, then v lies in the triangle abu because

$d(w) \geq 4$. Then either $\angle avu > 90°$ or $\angle bvu > 90°$, which is absurd. Similarly, bv is an edge and av is no edge.

If there is no further vertex, then u is adjacent to a, w, v, c, e. But then $d(v) \leq 4$, and another contradiction is obtained. Hence, there must exist an eighth vertex $w' \in$bd$Q$.

Assume first $w' \in bc$. By Remark ($\diamondsuit$), either eu or ev is an edge. Hence, w' is not adjacent to any point of ae. Since $d(w') \geq 4$ and by Remark ($\diamondsuit$), $w'u$ and $w'v$ are edges. Thus, ue and uc are edges, too. Again by Remark ($\diamondsuit$), wu and wv are edges. As $d(v) \geq 5$, there is a point $w'' \in wb \cup bw'$ adjacent to v. But then $d(w'') = 3$, absurd. Hence, there is no vertex $w' \in bc$. Analogously, there is no eighth vertex on ea.

Now we claim that there exists no edge zz' with $z \in ab \setminus \{a, b\}$, $z' \in ce \setminus \{c, e\}$. If, on the contrary, there are such edges, consider the one with z closest to a and z' closest to e. Since $d(u) \geq 5$, and since no further vertex lies on ae, u must be adjacent to some vertex on $az \cup ez'$ different from a, e, z, z'. But this vertex would have degree 3, a contradiction. The claim is proven.

Suppose there is an eighth vertex w' on ab. By Remark ($\diamondsuit$) and by the previous claim, neither w nor w' is adjacent to any point in $ae \cup ec \cup cb$. But $d(w) \geq 4$ and $d(w') \geq 4$, whence, both w, w' are joined to both u, v. Then either u is interior to a triangle $ww'v$, or v is interior to a triangle $ww'u$, in contradiction with the existence of the edges ua and vb.

Assume now the existence of an eighth vertex $w' \in ce$. We saw that no edge traverses int$Q$ from a point of ab to a point of ce, in particular ww' is not an edge of the triangulation. Since $d(w) \geq 4$ and $d(w') \geq 4$, wu, wv, $w'u$, $w'v$ are edges. Adding uv, this leads to the known acute triangulation of the square, of minimal size. We have to show that this is still impossible as a balanced triangulation.

Let D_u be the half-disc in Q of diameter ea and D_v be the half-disc in Q of diameter bc.

Let $p, q \in$ bdD_u, $r, s \in$ bdD_v be such that

$$\angle pae = \angle qea = \angle rcb = \angle sbc = 60°.$$

Let R be the rectangle $pqrs$.

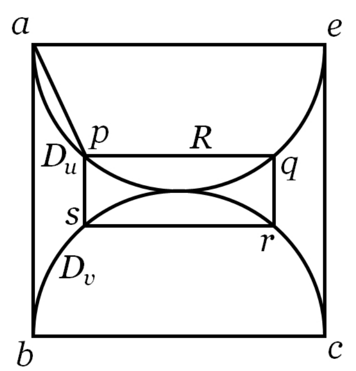

Fig. 3

Since $\angle bvc < 90°$ and $\angle aue < 90°$, $v \notin D_v$ and $u \notin D_u$. Furthermore, as $\angle bwv < 90°$ and $\angle awu < 90°$, the vertex u is farther away from bc than v. This and the inequalities $\angle wbv > 30°$, $w'cv > 30°$, $\angle wau > 30°$ and $\angle w'eu > 30°$ force $u, v \in R$.

Let p', s' be the orthogonal projections of p, s on ab, respectively.

Since $u, v \in R$ and both $\angle uwa$ and $\angle vwb$ are less than $90°$, $w \in p's'$. Now, if m is the midpoint of ab,

$$\angle uwv \le \angle pws \le \angle pms = 30°,$$

which is not allowed.

Hence, there must exist three vertices u, v, w in int$Q$.

Suppose there are no further vertices in int$Q$.

Assume, first, that u, v, w span a triangle Δ, and u, v, w lie in sinistrorsum order on Δ. Of course, no vertex is inside Δ. Let $u_1, ..., u_i$ be the neighbours of u outside Δ, let $v_1, ..., v_j$ be the neighbours of v outside Δ, and let $w_1, ..., w_k$ be the neighbours of w outside Δ, all in sinistrorsum order. Clearly, all these points belong to bd$Q$. Then $u_i = v_1$, $v_j = w_1$ and $w_k = u_1$. Since u, v, w have degree at least 5, we have $i \ge 3$, $j \ge 3$ and $k \ge 3$. Thus, u_2, v_2 and w_2 have only one neighbour among u, v, w. This means that they must be among the vertices a, b, c, e of Q. Assume without loss of generality, that $u_2 = a$, $v_2 = b$, $w_2 = c$. We observed that $d(e) = 3$. Then $u_3 = v_1 \in ab$, $v_3 = w_1 \in bc$, $w_3 = e$ and $w_4 = u_1 \in ea$, while $i = j = 3$ and $k = 4$. There are 11 triangles. Of this kind is also our example.

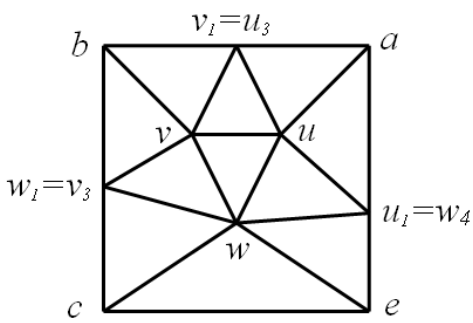

Fig. 4

Suppose now that uv, vw are edges, wu not. Let the neighbours of u be $v, u_1, ..., u_i$, those of v be $v_1 = u_i$, $v_2, ..., v_l, w, v_{l+1}, ..., v_j = u_1, u$, and those of w be $w_1 = v_l, w_2, ..., w_k = v_{l+1}, v$, all in sinistrorsum order.

Since u, v, w have degree at least 5, we have $i \ge 4$, $j \ge 3$, $k \ge 4$.

We have

$$d(u_1) \ge 4, \quad d(u_i) \ge 4, \quad d(u_2) = d(u_3) = 3.$$

Analogously,

$$d(w_1) \ge 4, \quad d(w_k) \ge 4, \quad d(w_2) = d(w_3) = 3.$$

Since boundary vertices have degree 3 only if they are vertices of Q, it follows that, say, $u_2 = e$, $u_3 = a$, $w_2 = b$, $w_3 = c$. It follows further that $i = k = 4$ and $3 \le j \le 4$.

The least number of boundary vertices, leading to the least number of triangles, is obtained for $j = 3$. Then $1 \leq l \leq 2$. In both cases there are 11 triangles. (If $j = 4$, then $l = 2$ and the number of triangles is 12.) So, in any case, we do not obtain fewer triangles than 11.

If the subgraph spanned by u, v, w is not connected, so, for example, if u is isolated, then there is an edge from one boundary neighbour p of u to another one, q. But pq cannot cut Q into a triangle and some other polygon, by Remark ($\diamondsuit$). Thus, pq cuts Q into two quadrilaterals, one containing u, the other v, w. Since u has at least 5 neighbours, all on bd$Q$, at least one of them is no vertex of Q and has degree 3, absurd.

We still have to treat the case of at least 4 vertices in int$Q$. We shall show that any such acute triangulation consists of at least 11 triangles.

Let $\alpha_0, \alpha_1, \alpha_2$ be the number of vertices, edges and faces, respectively. All faces but one are triangles. We see that $\alpha_0 \geq 9$. Indeed, otherwise the vertices must be a, b, c, e plus four in int$Q$. But the latter have degree at least 5, so each of them must have at least two neighbours in bd$Q$. As a, b, c, e can be such neighbours for only one interior vertex each, there must exist further boundary vertices.

Summing up the sides of all faces yields

$$(\alpha_2 - 1) \cdot 3 + 1 \cdot (\alpha_0 - 4) = 2\alpha_1,$$

whence $3\alpha_2 = 2\alpha_1 - \alpha_0 + 7$. Combining this with Euler's formula gives $\alpha_2 = \alpha_0 + 3 \geq 12$. Hence, the triangulation has at least 11 triangles. □

The sphere admits a balanced triangulation of size 20, and this is best possible. Indeed, Euler's formula – combined with the obvious condition that every vertex has degree at least 5 – yields the necessity of at least 20 triangles for any acute triangulation (see also [9]), and the regular icosahedron shows a realization (with all angles measuring 72°).

Other regular polygons

For $5 \leq n \leq 11$, the regular n-gon admits a balanced triangulation of size n with an extra vertex in its centre. For larger n, instead of treating here the rather complicated general case, we choose to present only a particular case, which displays a technique expected to be among those used in all cases.

Theorem 2. *The regular icosagon admits a balanced triangulation of size* 40.

Proof. Consider a 20-gon $v_1 ... v_{20}$ with centre o. Let $x_2 \in ov_2$ be chosen such that $\angle v_1 x_2 v_2 > 30°$, for example $\angle v_1 x_2 v_2 = 36°$. Then $|ox_2| = |v_1 x_2|$, and let x_i be the points analogously obtained ($i = 1, ..., 20$). We have $\angle v_1 o v_2 = 18°$, $\angle v_1 v_2 x_2 = 81°$ and $\angle v_2 v_1 x_2 = 63°$. The triangles ox_2x_4 and $v_3x_2x_4$ are congruent and have angles measuring 36°, 72°, 72°.

Thus, the triangles $v_{2i-1}v_{2i}x_{2i}$, $v_{2i+1}v_{2i}x_{2i}$, $ox_{2i}x_{2i+2}$ and $v_{2i+1}x_{2i}x_{2i+2}$ (where $i \in \{1, ..., 10\}$, indices mod. 20) form a balanced triangulation of size 40. □

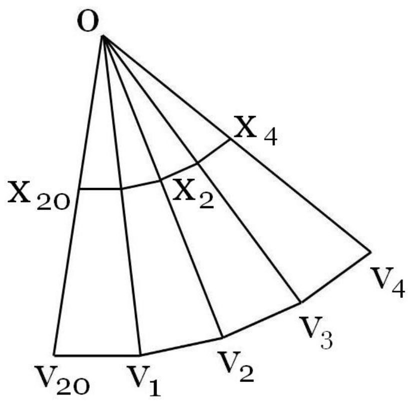

Fig. 5

Using this technique alone, we can construct balanced triangulations for all regular n-gons with $n = k \cdot 2^m$, where $k, m \in \mathbb{N}$ and $k \leq 11$. Thus, the first unresolved case is... the unlucky $n = 13$.

For the general case, the above technique does not suffice. We leave it open:

Problem. *Determine a lower bound for the size of a balanced triangulation of the regular n-gon.*

Moreover, we formulate the following.

Conjecture. *Every convex polygon with angles larger than* 30° *admits a balanced triangulation.*

The cube and the octahedron

We shall now provide balanced triangulations of all Platonic surfaces (i.e. boundaries of Platonic solids), all of them being optimal.

Theorem 3. *The boundary of the cube admits a balanced triangulation of size* 24*, and this is best possible.*

Proof. We use a construction from [8]: Fig. 6 exhibits a non-obtuse triangulation of the (unfolded) surface of the cube, using 24 triangles. Now, apply gentle shifts in the direction of the arrows, in the indicated order. As all angles remain or become acute, the only fact yet to be proved is that they are strictly larger than 30°. But as in the initial non-obtuse triangulation all angles were at least 45°, and like in the case of the square gentle shifts will not change this dramatically.

In [8] it is proved that this is best possible concerning acute triangulations. Thus, the minimality holds also for balanced triangulations. □

We remark here that at least four non-isomorphic balanced triangulations of the cube exist.

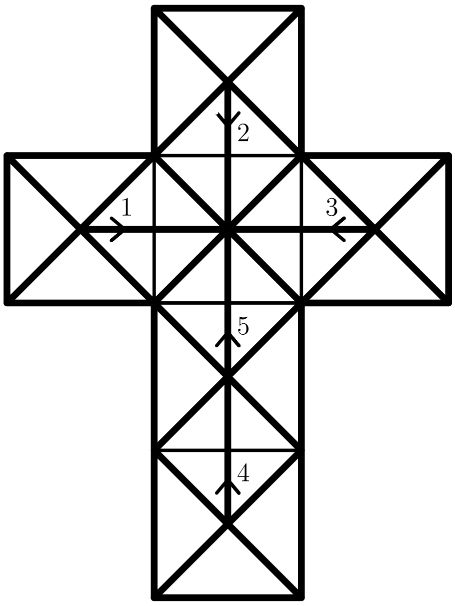

Fig. 6: A balanced triangulation of the surface of the cube

Theorem 4. *The regular octahedral surface admits a balanced triangulation of size* 8*, and this is best possible.*

Proof. The eight faces of the regular octahedron form a balanced triangulation. It remains to be proved that this is optimal. We show that there is no acute triangulation $\mathcal{T}$ of the octahedral surface with less than eight triangles. Employing a simple curvature argument, we know that each vertex of the regular octahedral surface has curvature $720°/6 = 120°$. This implies that no vertex of $\mathcal{T}$ may lie in the interior of a triangle of $\mathcal{T}$, whence, $\mathcal{T}$ must feature at least six vertices. Euler's formula now yields that there must be at least eight triangles in $\mathcal{T}$. □

The dodecahedron

In [11], Itoh and Zamfirescu investigated the triangulations of the regular dodecahedral surface. One conclusion was that the minimal size of a non-obtuse triangulation is 10. They also found that no acute triangulation with less than 12 triangles exists, and gave a triangulation with 14 acute triangles. But whether an acute triangulation with 12 triangles does or does not exist remained open. Here we present a triangulation with 12 acute triangles for the regular dodecahedral surface. Moreover, this triangulation is also balanced.

Lemma. *If an acute triangle Δ on the boundary surface of a regular dodecahedron D or regular icosahedron I contains in its interior a vertex of D or I, Δ must have all angles larger than* $36°$.

Proof. Clearly, each of D's (I's) 20 (12) vertices has curvature at least $36°$. Let Δ have angles α, β, and γ, and a vertex of D (or I) in its interior. We have

$\alpha + \beta + \gamma \geq 180^\circ + 36^\circ = 216^\circ$. As Δ is acute, any two of its angles sum up to less than 180°, whence α, β, and γ must each be greater than 36°, which proves the Lemma. □

Theorem 5. *The regular dodecahedral surface admits a balanced triangulation with* 12 *triangles.*

Proof. Assume the edge-length of the regular dodecahedron to be 1. We present now an acute triangulation with 12 triangles, see Fig. 7.

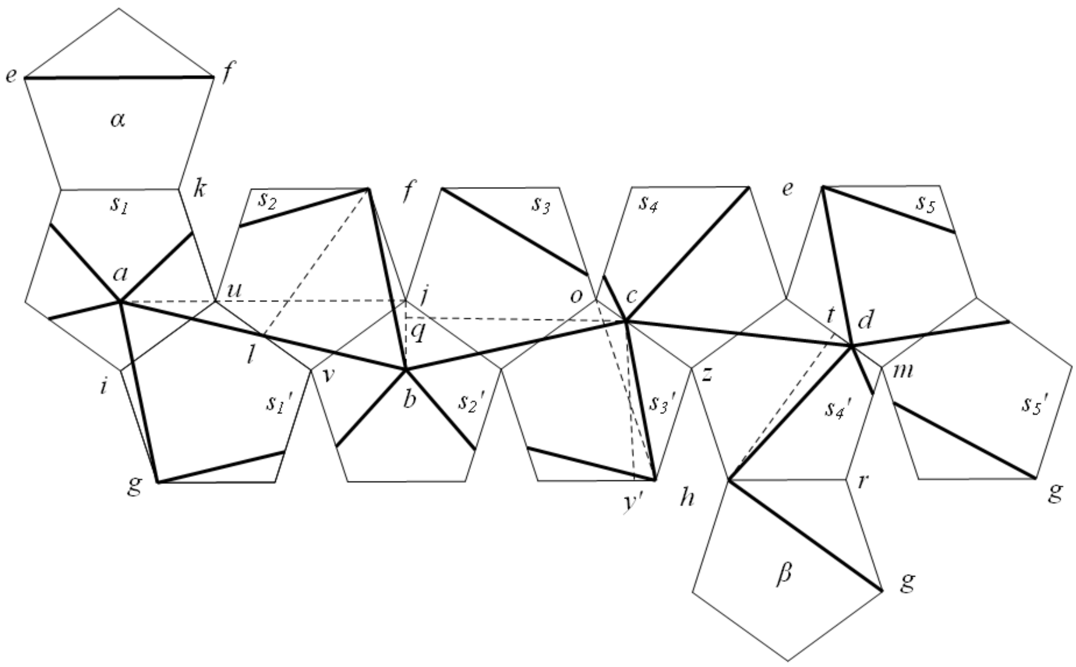

Fig. 7

Denote the upper and lower pentagonal faces of the dodecahedron by α and β, the five pentagons adjacent to α by s_1, s_2, s_3, s_4, s_5, and those adjacent to β by s'_1, s'_2, s'_3, s'_4, s'_5, so that s'_i, s_i, s_{i+1} have a common vertex, and s'_i, s'_{i+1}, s_{i+1} have a common vertex (indices taken modulo 5).

Let $\{i\} = s_1 \cap s'_1 \cap s'_5$, $\{j\} = s_2 \cap s_3 \cap s'_2$, $\{o\} = s_3 \cap s_4 \cap s'_3$, and $\{m\} = s'_4 \cap s'_5 \cap s_5$. Let a (resp. b) be the intersection point of the angular bisector of the angle at i (resp. j) in s_1 (resp. s'_2) and the diagonal of s_1 (resp. s'_2) determined by the two vertices adjacent to i (resp. j). Choose $c \in s'_3 \cap s_4$, $d \in s'_4 \cap s_5$ such that $|oc| = |dm| = \frac{5}{16}$. Put $\{e\} = \alpha \cap s_4 \cap s_5$, $\{f\} = \alpha \cap s_2 \cap s_3$, $\{g\} = \beta \cap s'_1 \cap s'_5$, and $\{h\} = \beta \cap s'_3 \cap s'_4$.

We get a triangulation $\mathcal{T}$ of the regular dodecahedral surface with 12 triangles:

$$aef, abf, bcf, cef, cde, ade, abg, bgh, bch, cdh, dhg, \text{ and } adg.$$

Noticing that there are two geodesics between a and g (resp. b and f), we use the one passing through s'_1 (resp. s_2).

We require the following trigonometric values.

$$\sin 18^\circ = \frac{\sqrt{5}-1}{4}, \quad \cos 18^\circ = \frac{\sqrt{10+2\sqrt{5}}}{4},$$
$$\sin 36^\circ = \frac{\sqrt{10-2\sqrt{5}}}{4}, \quad \text{and} \quad \cos 36^\circ = \frac{\sqrt{5}+1}{4}. \tag{1}$$

First part. $\mathcal{T}$ is a geodesic triangulation.

Here, we only need to show that the edges ab, bc and af are segments; for the others no proof is needed.

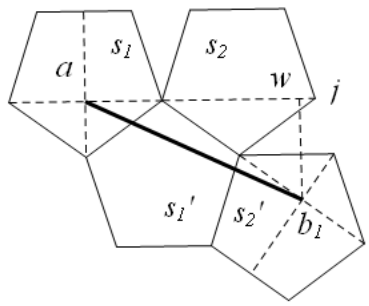

Fig. 8

In Fig. 7, the line-segment ab has the smallest length among all of the paths between a and b that pass through s_1, s_1', s_2 and s_2'. Furthermore, we have

$$|ab|^2 = |aj|^2 + |bj|^2 = (3\cos 36^\circ)^2 + \sin^2 36^\circ = \frac{9(3+\sqrt{5})}{8} + \frac{5-\sqrt{5}}{8} = 4+\sqrt{5}.$$

Now we consider the paths between a and b that pass through s_1, s_1', s_2' (for those passing through s_1, s_2, s_2', we are led to the same conclusion). Clearly the path ab_1, as shown in Fig. 8, has the shortest length. (For the sake of convenience, in this unfolding we denote b by b_1.) Now let w be the perpendicular projection of b_1 on aj. Then

$$\begin{aligned} |ab_1|^2 &= |aw|^2 + |b_1w|^2 \\ &= (\sin 36^\circ + \cos 36^\circ \sin 36^\circ)^2 + (2\cos 36^\circ + \cos^2 36^\circ)^2 \\ &= \frac{14+5\sqrt{5}}{4} > 4+\sqrt{5}. \end{aligned}$$

Hence, ab is indeed a segment.

Concerning bc, consider the orthogonal projection q of c on bj. Then

$$\begin{aligned} |bc|^2 &= |cq|^2 + |bq|^2 \\ &= \left(2\cos 36^\circ + \tfrac{5}{16}\cos 36^\circ\right)^2 + \left(\sin 36^\circ - \tfrac{5}{16}\sin 36^\circ\right)^2 \\ &= \frac{589+156\sqrt{5}}{256}. \end{aligned}$$

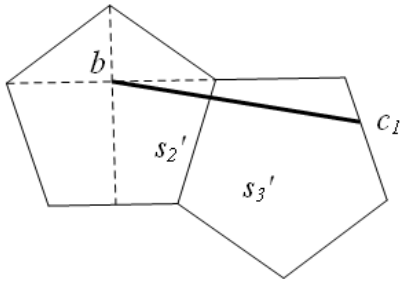

Fig. 9

Another short path between b and c goes through s_2' and s_3', so rotate s_3' around $s_2' \cap s_3'$ to become coplanar with s_2', and denote the new position of c by c_1, as shown in Fig. 9. We have

$$\begin{aligned} |bc_1|^2 &= \left(\cos 36^\circ + 1 + \tfrac{5}{16}\sin 18^\circ\right)^2 + \left(\tfrac{5}{16}\cos 18^\circ\right)^2 \\ &= \frac{505+200\sqrt{5}}{256} > \frac{589+156\sqrt{5}}{256}. \end{aligned}$$

Hence bc from Fig. 7 is indeed the segment from b to c.

Finally, in the unfolding of Fig. 7, denote the position of f in α by f_1. The line-segment af_1 (through s_1 and α) is longer than the segment af as shown in Fig. 7 (through s_1 and s_2), because, in the triangles akf and akf_1, $\angle a_1kf < \angle a_1kf_1$, where $\{k\} = \alpha \cap s_1 \cap s_2$. (See also Fig. 10, where a is denoted by a_1.) Hence af, too, as shown on Fig. 7, is indeed a segment.

Second part. $\mathcal{T}$ is an acute triangulation.

We start with the triangle aef. Since $\angle kfe = 72^\circ$ and $\angle lfk = 54^\circ$, where l is the midpoint of $s_1' \cap s_2$, to prove that $\angle afe < 90^\circ$ we just need to show that $\angle a_1fl > 36^\circ$ in Fig. 10. Indeed,

$$\tan \angle a_1fl = \frac{|a_1l|}{|fl|} = \frac{\frac{1}{2}+\cos 36^\circ}{\sin 36^\circ + \cos 18^\circ} > \tan 36^\circ = \frac{\sin 36^\circ}{\cos 36^\circ},$$

which is equivalent to

$$\frac{1}{2}\cos 36^\circ + \cos^2 36^\circ > \sin^2 36^\circ + \sin 36^\circ \cos 18^\circ.$$

Using (1), we get $\frac{2+\sqrt{5}}{4} > \frac{5+\sqrt{5}}{8}$, which is obviously true.

Denote the middle point of $s_1 \cap \alpha$ by w', the centre of s_2 by x, and $s_1 \cap s_2 \cap s_1'$ by $\{u\}$ in Fig. 10. Note that $\angle eaf = 2\angle w'a_1f = 2\angle a_1fl$. Since $|xl| > |lu|$ and $|fx| > |a_1u|$, we have $|fl| > |a_1l|$, which leads to $\angle a_1fl < 45^\circ$. Therefore $\angle eaf < 90^\circ$.

The triangle bgh is congruent to aef.

Let us now consider the triangle abf. We have $\angle abf < \angle vbj < 90^\circ$, where $\{v\} = s_1' \cap s_2 \cap s_2'$. Next, since $\angle a_1fl > 36^\circ$ which has been proved, $\angle fa_1u < 54^\circ$.

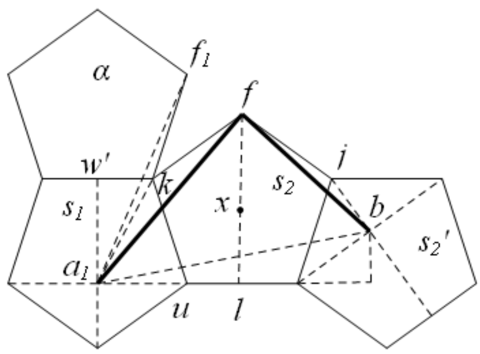

Fig. 10

In the triangle aul, $\angle ual + \angle ula = 36°$ and $|au| = |kj|/2 > |uv|/2 = |ul|$, so $\angle ual < 18°$. Then $\angle fab = \angle fau + \angle uab = \angle fa_1u + \angle uab < 72°$.

Finally,

$$|a_1f|^2 = \left(\frac{1}{2} + \cos 36°\right)^2 + (\sin 36° + \cos 18°)^2,$$

$$|bf|^2 = (\sin 18°)^2 + (\sin 36° + \cos 18°)^2$$

and

$$|a_1b|^2 = (1 + \cos 36° + \cos^2 36°)^2 + (\cos 36° \sin 36°)^2.$$

Using (1) and the fact that $\cos 36° \sin 36° = \frac{\cos 18°}{2}$, we get

$$|a_1f|^2 + |fb|^2 - |a_1b|^2 = \frac{1}{4} > 0.$$

Hence $\angle afb = \angle a_1fb < 90°$.

The triangle abg is congruent to abf.

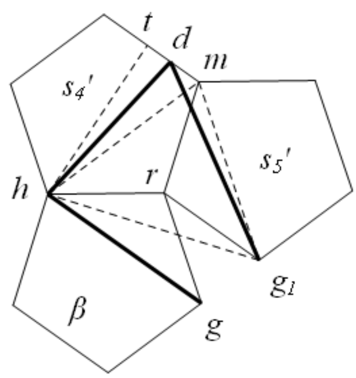

Fig. 11

Next consider the triangle dgh. Denote the midpoint of $s_4' \cap s_5$ by t and put $\{r\} = s_4' \cap s_5' \cap \beta$. Rotate s_5' around rm to become coplanar with s_4' as shown in Fig. 11, and let the new position of g in s_5' be g_1. Obviously, $\angle hgd < \angle hgm = 72°$ and $\angle dhg < \angle thg = 90°$. The circle passing through h, m, and g_1 has radius 1 and centre r. It has a tangent line at m perpendicular to rm. Since d lies outside this circle, $\angle hdg_1 < \angle hmg_1 = 72°$.

The triangle cef is congruent to dhg.

Now, take the triangle cdh into consideration. Let $\{z\} = s_3' \cap s_4 \cap s_4'$. The bisector of the angle ohz cuts oz in z', say. This bisector and zm are perpendicular in Fig. 7. Since $|zz'| < |zc|$ and $|md| < |zc|$, $\angle hcd < 90°$. Clearly, $\angle cdh < \angle tdh < 90°$. Finally, $\angle chd = \angle cht + \angle thd$. Since $\angle cht + \angle ohc = 90°$, it suffices to show $\angle thd < \angle cho$. Indeed,

$$\tan \angle thd = \frac{\frac{3}{16}}{\frac{1}{2}\tan 72°} = \frac{3\sin 18°}{8\cos 18°} < \tan \angle ohc = \frac{\frac{5}{16}\sin 36°}{2\cos 36° - \frac{5}{16}\cos 36°} = \frac{5\sin 36°}{27\cos 36°}$$

is equivalent to

$$40\sin 36°\cos 18° > 81\sin 18°\cos 36°.$$

Using (1) again, this reduces to $40\sqrt{5} > 81$.

The triangle cde is congruent to cdh.

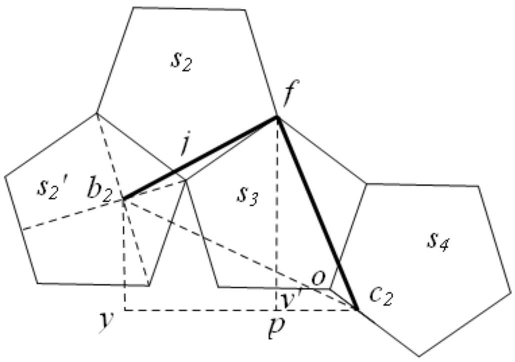

Fig. 12

Next, we turn our attention to the triangle bcf. In Fig. 7, rotate s_4 around o and s_2 around j to become adjacent with s_3 (see Fig. 12). Let v' be the midpoint of $s_3 \cap s_3'$ and denote the new position of c by c_2. The perpendicular from c_2 to the line fv' meets it at p. We have

$$\tan \angle v'fc_2 = \frac{|c_2p|}{|fv'| + |v'p|} = \frac{\frac{1}{2} + \frac{5}{16}\cos 36°}{\sin 36° + \cos 18° + \frac{5}{16}\sin 36°}.$$

Let q be the orthogonal projection of c on bj in the unfolding of Fig. 7. Observe that $\angle fcb = \angle fcq + \angle qcb$ where $\angle fcq = \angle fc_2p$, and $\angle fc_2p + \angle v'fc_2 = 90°$, so we just need to show $\angle qcb < \angle v'fc_2$. Indeed,

$$\tan \angle qcb = \frac{|bq|}{|cq|} = \frac{\frac{11}{16}\sin 36°}{(2 + \frac{5}{16})\cos 36°} = \frac{11\sin 36°}{37\cos 36°} < \tan \angle vfc_2 = \frac{\frac{1}{2} + \frac{5}{16}\cos 36°}{\cos 18° + \frac{21}{16}\sin 36°}$$

is equivalent to

$$11\sin 36°\cos 18° + \frac{231}{16}\sin^2 36° < \frac{37}{2}\cos 36° + \frac{185}{16}\cos^2 36°.$$

Using (1), this becomes $\frac{121\sqrt{5}+1155}{128} < \frac{777\sqrt{5}+1147}{128}$, which is correct.

Note that $\angle cbf = \angle cbq + \angle fbq$, and $\angle cbq + \angle bcq = 90°$, so we just need to check $\angle fbq < \angle bcq$. Indeed,

$$\tan \angle fbq = \frac{\sin 18°}{\frac{1}{2}\tan 72°} = \frac{2\sin^2 18°}{\cos 18°} < \tan \angle qcb = \frac{11 \sin 36°}{37 \cos 36°},$$

i.e.

$$74 \sin^2 18° \cos 36° < 11 \sin 36° \cos 18°,$$

which reduces to $15\sqrt{5} < 37$ by (1).

Denote the new position of b by b_2. We know that

$$|b_2 f|^2 = \sin^2 18° + (\sin 36° + \cos 18°)^2$$

and

$$|fc_2|^2 = \left(\frac{1}{2} + \frac{5}{16}\cos 36°\right)^2 + \left(\cos 18° + \frac{21}{16}\sin 36°\right)^2.$$

If y is the orthogonal projection of b_2 onto the line $c_2 p$, then

$$\begin{aligned} |b_2 c_2|^2 &= |b_2 y|^2 + |c_2 y|^2 \\ &= \left(\cos 36° \sin 18° + \tfrac{5}{16}\sin 36°\right)^2 \\ &\qquad + \left(\cos 36° \sin 18° + 2\sin 18° + 1 + \tfrac{5}{16}\cos 36°\right)^2. \end{aligned}$$

By (1), we get

$$|b_2 f|^2 + |c_2 f|^2 - |b_2 c_2|^2 > 0.$$

Therefore $\angle bfc = \angle b_2 f c_2 < 90°$.

The triangle adg is congruent to bfc.

Finally, we will show the triangle bch to be acute. It is easily seen that $\angle bhc < \angle y'hc^* < 90°$, where c^* is the mid-point of $s_3' \cap s_4$ and y' the orthogonal projection of c onto $s_3' \cap \beta$. Next, $\angle qcb < \angle jab$, for $\tan \angle jab = \frac{\sin 36°}{3\cos 36°} = \frac{1}{3}\tan 36°$ and $\tan \angle qcb = \frac{11}{37}\tan 36°$. So $\angle cbh < \angle abg < 90°$. Since $\angle bch = \angle bcy' + \angle y'ch$ and $\angle bcy' + \angle bcq = 90°$, it suffices to show that $\angle y'hc < \angle bcq$. Indeed

$$\tan \angle y'ch = \frac{\frac{1}{2} - \frac{5}{16}\cos 36°}{\sin 36° + \cos 18° - \frac{5}{16}\sin 36°} < \tan \angle bcq = \frac{11 \sin 36°}{37 \cos 36°},$$

because

$$296 \cos 36° - 185 \cos^2 36° < 176 \sin 36° \cos 18° + 121 \sin^2 36°,$$

which reduces via (1) to $22\sqrt{5} < 71$.

The triangle ade is acute, being congruent to bch.

Third part. $\mathcal{T}$ is a balanced triangulation.

Since each triangle of this triangulation has in its interior at least one vertex of the dodecahedron, by the Lemma, the triangulation is balanced. □

The icosahedron

Also for the regular icosahedral surface we find the same best possible lower bound, 12, for the size of a balanced triangulation.

Theorem 6. *The regular icosahedral surface admits a balanced triangulation of size* 12*, and this is best possible.*

Proof. We begin with the construction from [10], reproduced for the reader's convenience in Fig. 13. In [10] it is shown that this is an acute triangulation $\mathcal{T}$ of the icosahedral surface with 12 triangles, and that there exists none with fewer triangles, which implies that for a minimal balanced triangulation $\mathcal{T}^*$ of the regular icosahedral surface, we must have $\mathrm{card}\mathcal{T}^* \geq 12$.

We remark here that we cannot directly use the construction from Fig. 13, as $\angle cdb' < 30°$; for more details on this, see below.

First, we choose a planar embedding of the unfolding of the regular icosahedral surface as shown in Fig. 13, and we shall work on this unfolding throughout this proof. The point c is defined as orthogonal projection of a onto da'. This choice of c is the only (but crucial) modification with respect to the construction from [10], where c was at 1/4-th of the segment xy, closer to x.

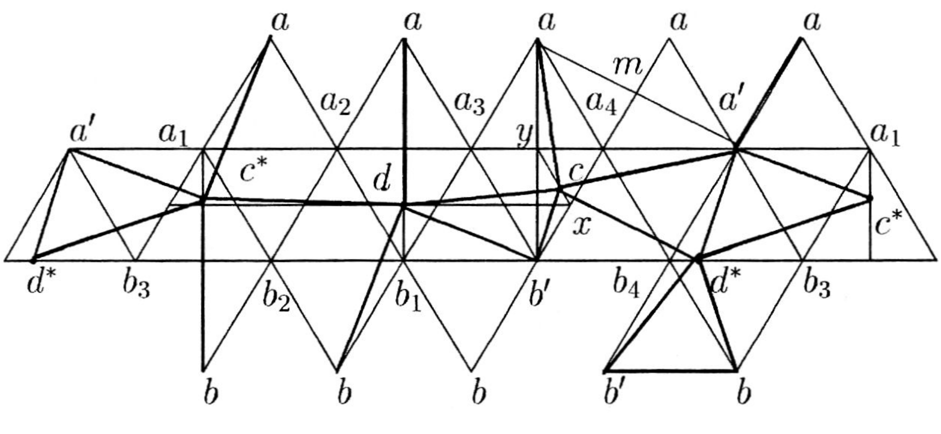

Fig. 13: The acute triangulation of the regular icosahedral surface featured in [10]

Let us motivate the change in c's position: the triangulation from [10] is not balanced. Indeed, notice that in [10], if we choose x to be the origin and the abscise-axis horizontal, then $c = \left(-\frac{1}{16}, \frac{\sqrt{3}}{16}\right)$, $d = \left(-\frac{5}{4}, 0\right)$, $b' = \left(-\frac{1}{4}, -\frac{\sqrt{3}}{4}\right)$. Thus,

$$\angle cdb' = \arccos \frac{\left\langle \left(\frac{19}{16}, \frac{\sqrt{3}}{16}\right), \left(1, \frac{\sqrt{3}}{4}\right) \right\rangle}{\left\| \left(\frac{19}{16}, \frac{\sqrt{3}}{16}\right) \right\| \cdot \left\| \left(1, \frac{\sqrt{3}}{4}\right) \right\|} = 28.62...° < 30°.$$

We will denote the point at c's old position as c^0. Let c' (the old c' will be called c'', see below) be the midpoint of the height at x of the triangle xa_4y. Furthermore,

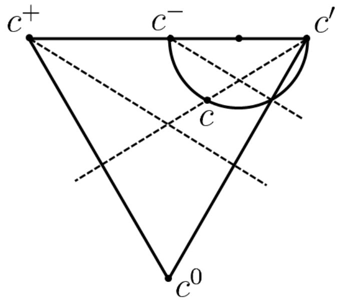

Fig. 14: c^0 (the old position of c), c', and c^+ form an equilateral triangle

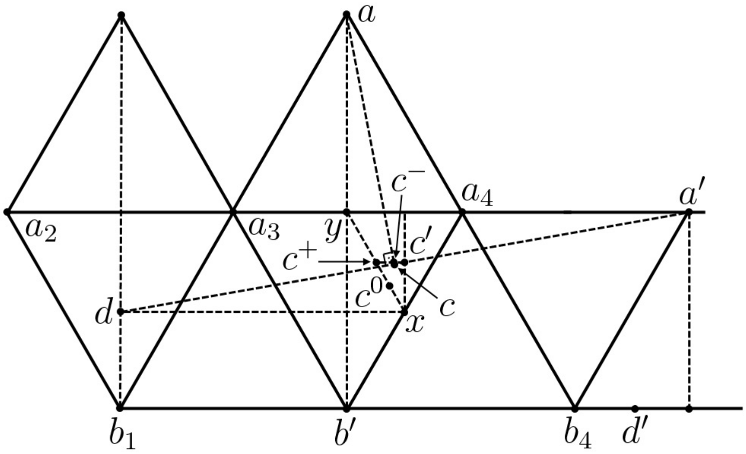

Fig. 15: ac and da' are perpendicular

let c^+ be the midpoint of xy. Notice that ac passes through the midpoint c^- of c^+c'. Notice also that da' itself passes through c'. See Figs. 14 and 15.

With the new position of c, we will now prove that all angles of the 12 triangles constituting $\mathcal{T}^*$, namely

$$aa'c, aa'c'', acd, ac''d, b'cd', a'c''d', bb'd, bb'd', b'cd, b'cd', bc''d, \text{and } bc''d',$$

are strictly larger than 30° and measure at most 90°. Finally, we shift points gently enough rendering all angles acute, while keeping every angle larger than 30°.

Note that both c^* and d^* are defined originally [10] as 'close' to c' and d', respectively. We will work in the following with the points c'' (this is the old c') and d' directly, and then provide appropriate shifts.

To emphasize on the c' versus c'' notation: Let c'' be now the midpoint of the line-segment joining the midpoints of a_1b_2 and a_1b_3 (note that in [10] this was c').

The triangles $aa'c''$, $ac''d$, $bb'd$, $bc''d$, and $bc''d'$ are balanced, the lower bound being given by the Lemma, and the upper bound by [10].

b′cd

$1^2 + (\frac{\sqrt{3}}{2} \cdot \frac{3}{2})^2 < \sqrt{3}^2$, whence d lies inside the circle $a'b'a_2$ with centre at a. On the greater arc $a'b'$ of this circle lies a_2, which sees $a'b'$ under the angle 30°. Hence d sees $a'b'$ under a larger angle, i.e. $\angle b'dc = \angle b'da' > 30°$.

We have $\angle b_1b'd > \angle yb'c'$, because $\tan \angle b_1b'd = \sqrt{3}/4 > \frac{1/4}{(3/4)\cdot(\sqrt{3}/2)} = \tan \angle yb'c'$. Hence $\angle db'c' < \angle b_1b'y = 90°$, and $\angle db'c < \angle db'c' < 90°$.

The other angles of $b'cd$ are easily treated:

$\angle b'cd < \angle b'qd = 90°$, where $\{q\} = ab' \cap dx$. Moreover, in $b'cd$, $|b'c| < |b'a_4| = |b'b_1| < |b'd|$, whence $\angle b'cd > \angle b'dc > 30°$.

b'd'c

We first show that $\angle b'cd' < 90°$. As c^+ is the midpoint of the segment xy, the line-segments b_4x and $d'c^+$ are parallel. Hence, if $\{z\} = b'a_4 \cap c^+d'$, we have $\angle b'cd' < \angle b'zd' = 90°$.

We show now that $\angle b'd'c < 90°$. Since the line-segments b_4x and $d'c^+$ are parallel, $\angle b_4d'c^+ = 30°$. We have

$$\angle b'd'c = \angle b'd'b_4 + \angle b_4d'c = \angle a_4d'b_4 + \angle b_4d'c = (60° - \angle b_4a_4d') + (30° + \angle c^+d^-c).$$

Thus, it suffices to show that $\angle c^+d'c < \angle b_4a_4d'$.

Recall that q is the midpoint of $b'y$. The centre u of the circle Γ through c^+, c^-, q lies on the bisectors of c^+c^- and c^+q. Hence, its radius is shorter than xc^+, which has length $1/4$. The distance from d' to u is far larger, whence d' lies outside Γ and $\angle c^+d'c^- < \angle c^+qc^-$. Hence

$$\angle c^+d'c < \angle c^+d'c^- < \angle c^+qc^- = \angle c^+yc^- = \angle b_4a_4d'.$$

We prove now that $\angle cb'd' < 90°$. It is enough to show $\angle b_4b'd' < \angle yb'c$. Indeed, $\angle b_4b'd' = \angle b_4a_4d' = \angle c^+yc^-$, $\tan \angle b_4b'd' = \sqrt{3}/8$ and $\tan \angle yb'c^- = \frac{(3/4)\cdot(1/4)}{(3/4)\cdot(\sqrt{3}/2)} = \sqrt{3}/6$. Hence $\angle b_4b'd' < \angle yb'c^- < \angle yb'c$.

Via the Lemma, all angles of the triangle $b'd'c$ measure more than 30° (as b_4 is in the interior of $b'd'c$).

d'ca'

First let us show that $\angle d'ca' > 30°$. Indeed, in the small triangle $c^0c'c^+$ (see Fig. 14), the circle of diameter $c'c^-$ does not meet c^+d' (which is orthogonal on c^0c'). Since c lies on that circle, $\angle d'ca' > \angle d'c^+a' > \angle d'c^+c' = 30°$.

The angle $a'd'c$ is acute because $\angle a'd'c < \angle a'd'c^+$ and $\angle a'd'c^+ < 90°$ since $d'c^+$ and $a'b_4$ are orthogonal. Moreover, $\angle d'a'c > \angle b_4a'b' = 30°$ and $\angle d'a'c < \angle ba'a_4 = 90°$.

bb'd'

We find the proofs for all required upper bounds of the angles occurring in the triangle $bb'd'$ in [10]. For the lower bounds, we have $\angle bb'd^* > \angle bb'd_3 = 30°$, $\angle b'bd^* > \angle b'bb_4 = 60°$, and $\angle bd^*b' > \angle bb_3b' = 30°$.

a'c''d'

The upper bounds for the angles $\angle a'd'c''$ and $\angle d'c''a'$ are supplied by [10], and $\angle d'a'c'' = 90°$ by construction. Furthermore, we have $\angle d'c''a' > \angle a'a_1d' > \angle a'a_1b_4 = 30°$ and $\angle a'd'c'' > \angle a'b_4c'' > \angle a'b_4a_1 = 30°$.

acd

All angles of acd are greater than 30° due to the Lemma (a_3 is in the interior of the triangle acd). By construction, $\angle dca = 90°$. We have $\angle cad = \angle a_3ad + \angle a_3ay + \angle yac < 90°$, as $\angle a_3ad = \angle a_3ay = 30°$ and $\angle yac < \angle yaa_4 = 30°$. Lastly, obviously we have $\angle cda < \angle adx = 90°$.

aa'c

In $aa'c$, all angles are greater than $30°$ due to the Lemma (which is applicable as a_4 lies in the interior of $aa'c$). We have $\angle aa'c < 90°$, as $\angle a_4a'c < \angle a_4a'b' = 30°$, and $\angle caa' < 90°$, as $\angle caa_4 < \angle yaa_4 = 30°$. Finally, $\angle a'ca = 90°$ by construction (see Fig. 15).

It remains to shift a little some of the vertices of this triangulation in order to render it acute. Concretely, c'' will be slightly shifted towards a_1, and c will be shifted away from a. □

Unbounded geometry

As long as we remain confined to the Euclidean plane, the minimal size of balanced triangulations – unlike acute triangulations – will always depend on the ratio width/diameter.

Theorem 7. *Let $\mathcal{F}$ be a family of polygons in $\mathbb{R}^2$ such that the infimum of all ratios width/diameter vanishes. Then there is no number N for which every polygon in $\mathcal{F}$ admits a balanced triangulation of size at most N.*

Proof. Choose arbitrarily a natural number N. Let $P \in \mathcal{F}$ have ratio width/diameter less than $1/(2N)$. Then its diameter is larger than $2Nw$, where w is its width.

Suppose that P admits a balanced triangulation of size N. Let $Q \subset P$ be a shortest path from a to b, where $a, b \in P$ are such that $|ab|$ is the diameter of P. Let $T_0, ..., T_n$ be the ordered finite sequence of triangles met by Q from $a \in T_0$ to $b \in T_n$. Choose $p_i \in Q \cap T_i$ $(0 \le i \le n)$. Between p_{i-1} and p_i there is some point $q_i \in Q \cap T_{i-1} \cap T_i$ $(1 \le i \le n)$.

In every triangle of a balanced triangulation the ratio width/diameter is larger than $2/\sqrt{3} > 1/2$. In T_i, $|q_iq_{i+1}| < \Delta_i$ and $w_i < w$, where Δ_i and w_i are the diameter and width of T_i, respectively $(1 \le i \le n-1)$. Hence

$$\frac{|q_iq_{i+1}|}{w} < \frac{\Delta_i}{w_i} < 2.$$

Adding the triangles T_0 and T_n, and summing up,

$$|ab| \le |aq_1| + \left(\sum_{i=1}^{n-1} |q_iq_{i+1}|\right) + |q_nb| < 2(n+1)w \le 2Nw.$$

This contradiction ends the proof. □

On arbitrary surfaces the situation may change, as shown by the next example.

Example. Let Z be the surface of a right bounded circular cylinder (the boundary of the cartesian product of a line-segment with a circular disc). While the infimum of the ratio width/diameter vanishes for the family of all Z, we can always find a balanced triangulation with 20 triangles.

Indeed, let D_1 and D_2 be the two discs in Z. Take a pentagon $P_i \subset D_i$ concentric with D_i. Choose P_1 and P_2 such that the smallest angle between a side of P_1 and a side of P_2 be of 36°. Let $P_i = a_i b_i c_i d_i e_i$. Arrange that the orthogonal projection a_2' of a_2 onto D_1 lies between a_1 and b_1 on the circle containing $a_1, b_1, c_1, d_1, e_1, a_2', b_2', c_2', d_2'$ and e_2'. The isosceles geodesic triangle $a_1a_2b_1$ has a total curvature of 36°. If P_1 and P_2 are small, then $\angle a_1a_2b_1$ is small (close to 0°). If P_1 and P_2 are large (a_1 comes close to the boundary of D_1), then $\angle a_1a_2b_1$ is large (close to 162°). Thus, there is a convenient position of P_1 and P_2 for which $\angle a_1a_2b_1 = 40°$. Then the other two angles of the (geodesic) triangle $a_1a_2b_1$ have together 174°, whence, the triangle is balanced. There are 10 triangles congruent to $a_1a_2b_1$, and another 10 trivially obtained inside of P_1 and P_2. This triangulation is balanced.

Similar examples are right prisms over regular polygons with more than 4 sides, and any convex surfaces close to these examples with respect to the Pompeiu-Hausdorff distance.

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References

[1] A. D. Alexandrov. On partitions and tessellations of the plane (Russian), *Mat. Sbornik* **2** (1937) 307–318.

[2] B. S. Baker, E. Grosse, and C. S. Rafferty. Nonobtuse triangulations of polygons, *Discrete Comput. Geom.* **3** (1988) 147–168.

[3] M. Bern, S. Mitchell, and J. Ruppert. Linear-size nonobtuse triangulations of polygons, *Discrete Comput. Geom.* **14** (1995) 411–428.

[4] Ch. Cassidy and G. Lord. A square acutely triangulated, *J. Recr. Math.* **13** (1980–81) 263–268.

[5] D. Eppstein. `http://www.ics.uci.edu/~eppstein/junkyard/acute-square`

[6] C. Fulton. Tessellations, *Am. Math. Monthly* **99** (1992) 442–445.

[7] J. L. Gerver. The dissection of a polygon into nearly equilateral triangles, *Geom. Dedicata* **16** (1984) 93–106.

[8] T. Hangan, J. Itoh, and T. Zamfirescu. Acute triangulations, *Bull. Math. Soc. Sci. Math. Roum.* **43** (91) (2000) 279–285.

[9] J. Itoh. Acute triangulations of sphere and icosahedron, *Josai Math. Monogr.* **3** (2001) 53–62.

[10] J. Itoh and T. Zamfirescu. Acute triangulations of the regular icosahedral surface, *Discrete Comput. Geom.* **31** (2004) 197–206.

[11] J. Itoh and T. Zamfirescu. Acute triangulations of the regular dodecahedral surface, *Europ. J. Combin.* **28** (2007) 1072–1086.

[12] I. Niven. Convex polygons that cannot tile the plane, *Am. Math. Monthly* **85** (1978) 785–792.

[13] L. Yuan. Acute triangulations of polygons, *Discrete Comput. Geom.* **34** (2005) 697–706.

[14] C. T. Zamfirescu. Survey of two-dimensional acute triangulations, *Discrete Math.* **313** (2013) 35–49.

[15] T. Zamfirescu. Acute triangulations: a short survey. *Proc. 6th Ann. Conf. Roum. Soc. Math. Sci.* **I** (2002) 10–18.

Lijie JIA, Liping YUAN

College of Mathematics and Information Science,
Hebei Normal University, 050024 *Shijiazhuang,*
P.R. China.

E-mail address: *lpyuan@mail.hebtu.edu.cn*

Carol T. ZAMFIRESCU

Fakultät für Mathematik,
Technische Universität Dortmund,
Germany

E-mail address: *czamfirescu@gmail.com*

Tudor I. ZAMFIRESCU

Fakultät für Mathematik,
Technische Universität Dortmund,
Germany

and

"Simion Stoilow" Institute of Mathematics,
Roumanian Academy,
Bucharest, Roumania

and

"Abdus Salam" School of Mathematical Sciences,
GC University,
Lahore, Pakistan.

E-mail address: *tuzamfirescu@gmail.com*