Lattice graphs with non-concurrent longest cycles

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ABSTRACT – No hypohamiltonian graphs are embeddable in the planar square lattice. This lattice contains, however, graphs in which every vertex is missed by some longest cycle. In this paper we present graphs with this property, embeddable in various lattices, and of remarkably small order.

1. Introduction

If you enter a warehouse and want to see a maximal number of items among a given set of interesting items, but dislike seeing any item twice, you may have several optimal options. Does an item exist, which you will have to see – independent of the optimal solution you choose?

This "practical" question leads to the graph-theoretical problem treated here.

Hypohamiltonian graphs have attracted the interest of mathematicians, not only of those working in Graph Theory, since the early discovery by A. Kempe [6]

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of the well-known Petersen graph [12]. In fact, in the beginning, this graph was not yet investigated in relation with its hypohamiltonicity. A graph is *hypohamiltonian* if it is not hamiltonian, i.e. has no spanning cycle, but deprived of any vertex becomes hamiltonian.

The *hypotraceable* graphs, analogously defined by replacing "cycle" with "path", needed much longer to appear. Before the first hypotraceable graphs were found, T. Gallai asked about the existence of finite graphs in which any vertex is missed by some longest path [3], a condition weaker than hypotraceability. Shortly thereafter, H. Walther found such graphs, even planar ones [15]. Then the question of minimality of their order was asked (in [17]), and this question makes sense for the case of cycles instead of paths as well.

The question was also diversified by asking several arbitrary vertices to be missed, by separately treating the planar case, and by increasing the connectivity. A detailed survey of the results obtained along these lines is [13]. (See also [19].)

Later these questions were asked with respect to finite graphs embeddable in the various existing lattices (by the third author). Here, a graph is *embedabble* in a lattice if it can be realized as a subgraph of that lattice. The first such lattice, $\mathbb{L}_n$, with $\mathbb{Z}^n$ as vertex set and with an edge (x, y) whenever the Euclidean distance between x and y (in $\mathbb{R}^n$) is 1, was considered by B. Menke in [7]. While Menke himself as well as Menke, Ch. Zamfirescu and the third author [8] treated a special subfamily of the family $\mathcal{L}^2$ of all finite 2-connected subgraphs of $\mathbb{L}_2$, the so-called "grid graphs", with the outcome that they contain no graph with empty intersection of all of its longest cycles, in [7] also appears a graph in $\mathcal{L}^2$ found earlier by G. Wegner, of order 95, in which every vertex is missed by a suitable longest cycle. For twenty years no smaller example has been exhibited!

Results concerning fault-tolerant designs in computer networks (for a selection of articles on fault-tolerance problems in Graph Theory, see [4], [16], [5], [2], [10], [11]) further motivate studying these questions in lattices.

Let $\mathcal{G}$ be the family of all finite connected graphs in which every vertex is missed by some longest cycle. It is our goal in this paper to provide graphs in $\mathcal{G}$ of much smaller order than the ones known so far, embeddable in the square, triangular, hexagonal, and cubic lattices, which we denote by $\mathbb{L}_2$, $\mathbb{T}$, $\mathbb{H}$, $\mathbb{L}_3$, respectively.

Let $\mathcal{L}^n$, $\mathcal{T}$, $\mathcal{H}$ be the families of all finite 2-connected subgraphs of $\mathbb{L}_n$ ($n = 2, 3$), $\mathbb{T}$, $\mathbb{H}$, respectively. Notice that $\mathcal{H} \subset \mathcal{L}^2 \subset \mathcal{T}$, so any graph G embeddable into $\mathbb{H}$ will always be embeddable into $\mathbb{L}_2$, and also embeddable into $\mathbb{T}$. Although $\mathbb{T}$ and $\mathbb{L}_3$ are both 6-regular, there exist graphs embeddable into $\mathbb{T}$ but not into $\mathbb{L}_3$, and vice-versa. Observe that there are no 3-connected finite subgraphs of $\mathbb{L}_2$ (and, hence, neither of $\mathbb{H}$).

2. Small graphs in $\mathcal{G} \cap \mathcal{L}^2$ and $\mathcal{G} \cap \mathcal{L}^3$

We exhibit here a new graph, homeomorphic to C. Thomassen's example T with 15 vertices which first appeared in [18, p. 216], and show that it also belongs to $\mathcal{G}$. It will be very useful towards our stated goals.
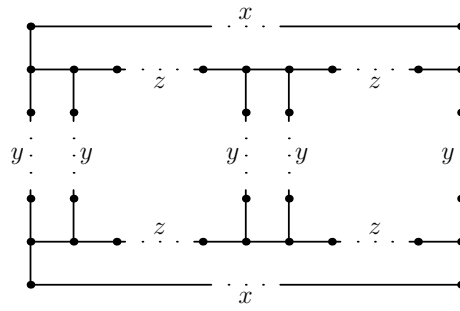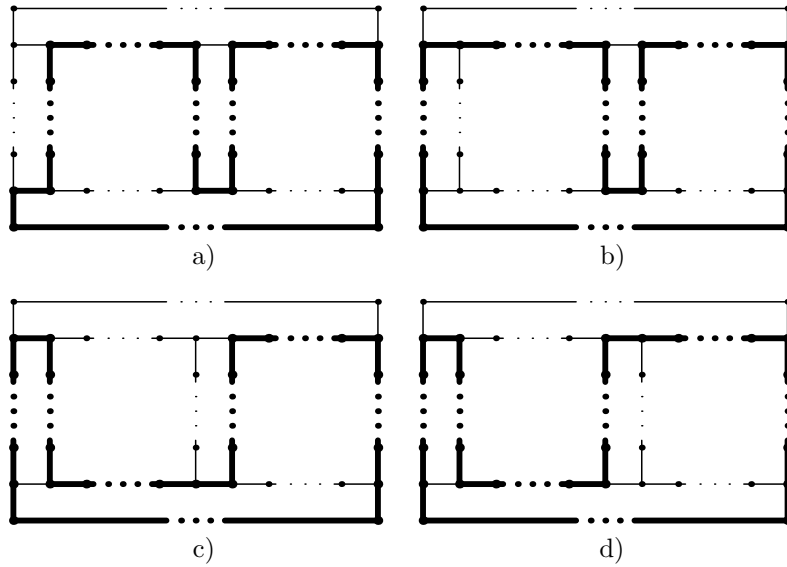
Fig. 1

Let $G_{x,y,z}$ be the graph in Fig. 1, where x, y, z denote the numbers of vertices of degree 2 on the respective paths.

LEMMA 2.1. $G_{x,y,z}$ belongs to $\mathcal{G}$ if and only if $2y \geq x + 2z + 1$.

Figs. 2 a) – d)

PROOF. There are three types of long cycles in G. The cycles in Figs. 2 a)–e) all have length $9 + x + 4y + 2z$. The cycles in Figs. 2 f)–g) have length $10 + 2x + 2y + 4z$. The cycle in Fig. 2 h) has length $10 + 2x + 2y + 2z$. Those in Figs. 2 a)–e) have to be longest in order for $G_{x,y,z} \in \mathcal{G}$ to hold, and vice-versa. Hence, this holds if

and only if $9 + x + 4y + 2z \geq 10 + 2x + 2y + 4z$, which is the inequality of the
statement. $\qquad\square$
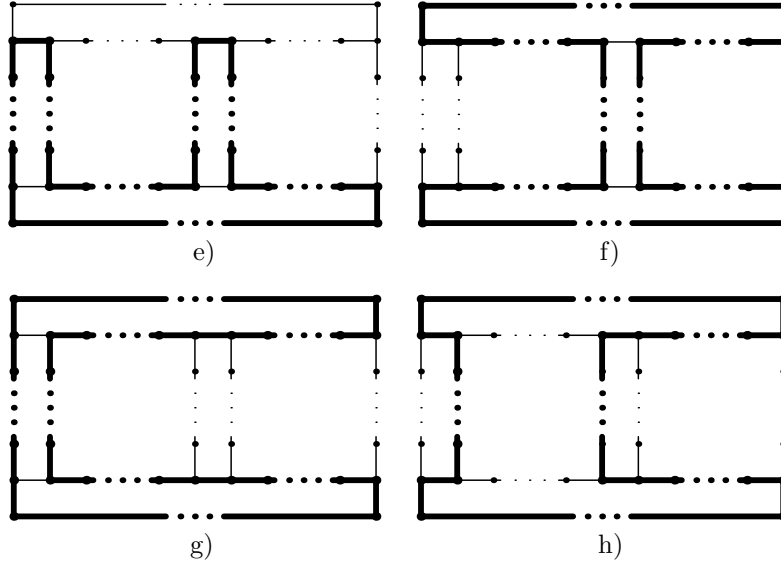
Figs. 2 e) – h)

Our Lemma enables us to construct a new graph belonging to $\mathcal{G} \cap \mathcal{L}^2$, greatly
improving upon the previous smallest known bound 95.

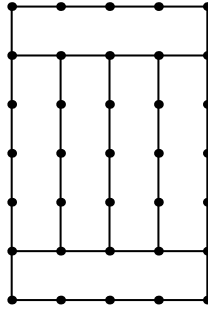THEOREM 2.2. *There exists a graph in $\mathcal{G} \cap \mathcal{L}^2$ of order* 35.

Fig. 3

PROOF. Fig. 3 shows an embedding of $G_{5,3,0}$ in $\mathbb{L}_2$. $\qquad\square$

In [1] two different graphs homeomorphic to Petersen's graph are used to show that $\mathcal{G} \cap \mathcal{L}^3$ contains a graph of order 40 and $\mathcal{G} \cap \mathcal{L}^4$ one of order 20. We now improve this as follows.

THEOREM 2.3. *There exists a planar graph in $\mathcal{G} \cap \mathcal{L}^n$ of order* 17, *for any* $n \geq 3$.
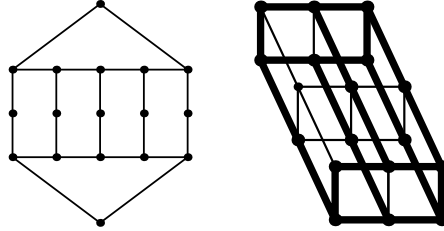

Fig. 4: The graph $G_{1,1,0}$ and its isomorphic embedding into $\mathcal{L}^3$

PROOF. Indeed, the graph $G_{1,1,0}$ can be embedded in $\mathbb{L}_3$, see Fig. 4. $\qquad \square$

3. Small graphs in $\mathcal{G} \cap \mathcal{T}$ and $\mathcal{G} \cap \mathcal{H}$

A. Shabbir and the third author [14] proved that $\mathcal{G} \cap \mathcal{T}$ contains a graph on 60 vertices. Our Lemma shows again its power and leads to the following.

THEOREM 3.1. *There exists a graph in $\mathcal{G} \cap \mathcal{T}$ of order* 33.

PROOF. Clearly, the graph presented in Fig. 5 (homeomorphic to the one used in [14] but much smaller) lies in $\mathcal{T}$. It satisfies the inequality from the Lemma, with $x = 4$, $y = 3$ and $z = 0$, and is therefore in $\mathcal{G}$. $\qquad \square$
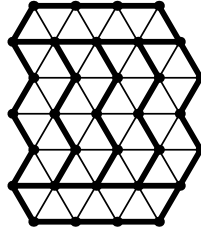

Fig. 5

The first graph in $\mathcal{G} \cap \mathcal{H}$, of order 170, has been uncovered by F. Nadeem, Shabbir, and the third author [9].

THEOREM 3.2. *There exists a graph in* $\mathcal{G} \cap \mathcal{H}$ *of order* 89.

PROOF. By the Lemma, $G_{13,9,2} \in \mathcal{G}$. An embedding of $G_{13,9,2}$ in $\mathbb{H}$ and all its
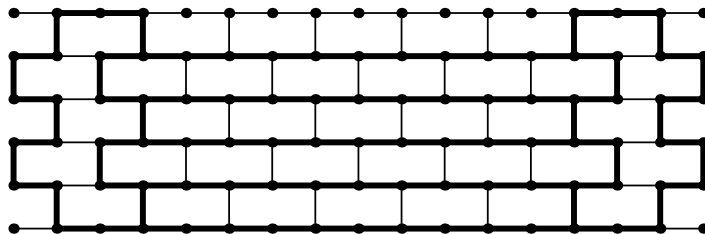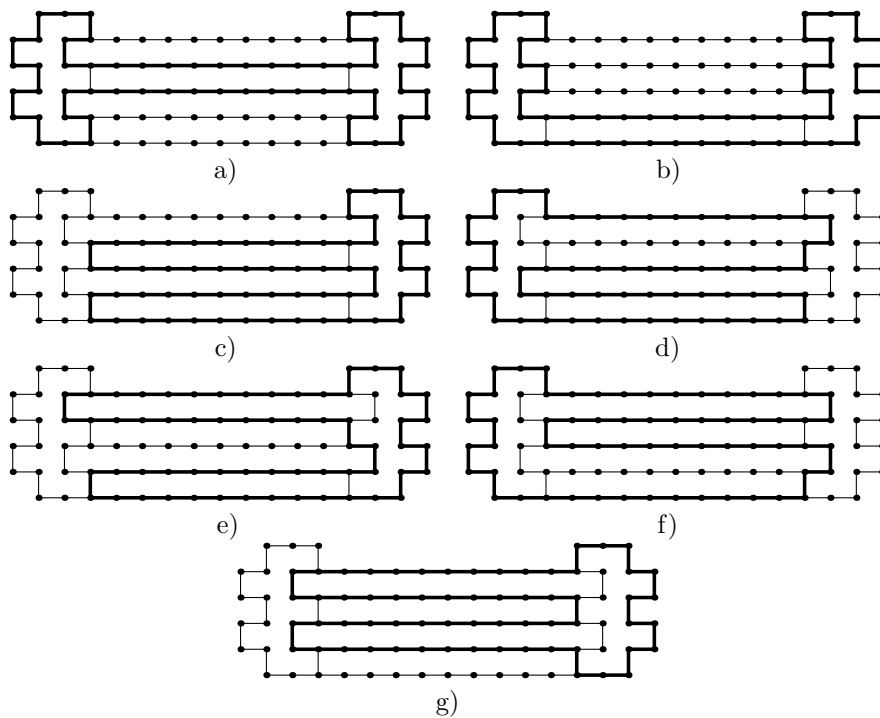longest cycles are shown in Figs. 6 and 7.   □

Fig. 6

a)

b)

c)

d)

e)

f)

g)

Figs. 7 a) – g)

4. Open questions

Our work leaves open the existence of 3-connected examples. Let $\mathcal{G}_3$ be the set of all 3-connected graphs in $\mathcal{G}$.

Question 1. Does $\mathcal{G}_3 \cap \mathcal{T} = \emptyset$ hold?

Question 2. Does $\mathcal{G}_3 \cap \mathcal{L}^n = \emptyset$ hold, for each $n \geq 3$?

References

[1] Y. Bashir – T. Zamfirescu, *Lattice graphs with Gallai's property*, Bull. Math. Soc. Sci. Math. Roumanie, **56** (2013), pp. 65–71.

[2] Y.-C. Chen – C.-H. Tsai – L.-H. Hsu – J. J. M. Tan, *On some super fault-tolerant Hamiltonian graphs*, Appl. Math. Comput., **148** (2004), pp. 729–741.

[3] T. Gallai, *Problem 4*, in: *Theory of Graphs Proc. Tihany* 1966 (eds.: P. Erdös & G. Katona), Academic Press, New York, 1968, p. 362.

[4] J. P. Hayes, *A Graph Model for Fault-Tolerant Computing Systems*, IEEE Trans. Comput., **25** (1976), pp. 875–884.

[5] C.-N. Hung – L.-H. Hsu – T.-Y. Sung, *On the Construction of Combined k-Fault-Tolerant Hamiltonian Graphs*, Networks, **37** (2001), pp. 165–170.

[6] A. B. Kempe, *A Memoir on the Theory of Mathematical Form*, Phil. Trans. Roy. Soc. Lond., **177** (1886), pp. 1–70.

[7] B. Menke, *Longest cycles in grid graphs*, Studia Sci. Math. Hung., **36** (2000), pp. 201–230.

[8] B. Menke – Ch. Zamfirescu – T. Zamfirescu, *Intersections of longest cycles in grid graphs*, J. Graph Theory, **25** (1997), pp. 37–52.

[9] F. Nadeem – A. Shabbir – T. Zamfirescu, *Planar lattice graphs with Gallai's property*, Graphs Combin., DOI: 10.1007/s00373-012-1177-8.

[10] J.-H. Park – H.-C. Kim – H.-S. Lim, *Fault-hamiltonicity of hypercube-like interconnection networks*, Proc. of IEEE International Parallel and Distributed Processing Symposium IPDPS 2005, Denver, Apr. 2005.

[11] J.-H. Park – H.-S. Lim – H.-C. Kim, *Panconnectivity and pancyclicity of hypercube-like interconnection networks with faulty elements*, Theoret. Comput. Sci., **377** (2007), pp. 170–180.

[12] J. Petersen, *Sur le théorème de Tait*, L'Intermédiaire des Mathématiciens, **5** (1898), pp. 225–227.

[13] A. Shabbir – C. T. Zamfirescu – T. I. Zamfirescu, *Intersecting longest paths and longest cycles: A survey*, Electron. J. Graph Theory Appl., **1** (2013), pp. 56–76.

[14] A. Shabbir – T. Zamfirescu, *Fault-tolerant designs in triangular lattice networks*, submitted.

[15] H. Walther, *Über die Nichtexistenz eines Knotenpunktes, durch den alle längsten Wege eines Graphen gehen*, J. Combin. Theory, **6** (1969), pp. 1–6.

[16] T. Yamada – S. Ueno, *Fault-tolerant graphs for tori*, Networks, **32** (1998), pp. 181–188.

[17] T. Zamfirescu, *A two-connected planar graph without concurrent longest paths*, J. Combin. Theory, Ser. B, **13** (1972), pp. 116–121.

[18] T. Zamfirescu, *On longest paths and circuits in graphs*, Math. Scand., **38** (1976), pp. 211–239.

[19] T. Zamfirescu, *Intersecting longest paths or cycles: a short survey*, Analele Univ. Craiova, Ser. Mat.-Inf., **28** (2001), pp. 1–9.