Hamiltonian Connectedness of Toeplitz Graphs

Muhammad Faisal Nadeem, Ayesha Shabbir, and Tudor Zamfirescu

1 Introduction

A simple undirected graph T with vertices $1, 2, \ldots, n$ is called a *Toeplitz graph* if its adjacency matrix $A(T)$ is Toeplitz. A *Toeplitz matrix* is an $(n \times n)$ symmetric matrix which has constant values along all diagonals parallel to the main diagonal. Therefore, a Toeplitz graph T is uniquely defined by the first row of $A(T)$, a (0–1) sequence. If the 1's in that sequence are placed at positions $1 + t_1$, $1 + t_2, \ldots, 1 + t_k$ with $1 \leq t_1 < t_2 < \cdots < t_k < n$, we may simply write $T = T_n \langle t_1, t_2, \ldots, t_k \rangle$, two vertices x, y being connected by an edge iff $|x - y| \in \{t_1, t_2, \ldots, t_k\}$.

Let G be a graph of order n. It is called *Hamiltonian* if it contains a cycle of order n. It is called *traceable*, if it contains a path of order n; that path is then called a *Hamiltonian path* of G. The graph G is said to be *Hamiltonian connected* if for any pair of distinct vertices u and v of G, there exists a Hamiltonian path from u to v. The property of being Hamiltonian connected is stronger than being Hamiltonian.

The third author's work was supported by a grant of the Romanian National Authority for Scientific Research, CNCS – UEFISCDI, project number PN-II-ID-PCE-2011-3-0533.

M.F. Nadeem • A. Shabbir
Abdus Salam School of Mathematical Sciences, GC University, 68-B, New Muslim Town, Lahore, Pakistan
e-mail: mfaisalnadeem@ymail.com; ashinori@hotmail.com

T. Zamfirescu (✉)
Faculty of Mathematics, University of Dortmund, 44221 Dortmund, Germany

Institute of Mathematics "Simion Stoïlow" Roumanian Academy, Bucharest, Roumania

Abdus Salam School of Mathematical Sciences, GC University, 68-B, New Muslim Town, Lahore, Pakistan
e-mail: tuzamfirescu@googlemail.com.

P. Cartier et al. (eds.), *Mathematics in the 21st Century*, Springer Proceedings in Mathematics & Statistics 98, DOI 10.1007/978-3-0348-0859-0_8

References [1–4] contain results about connectivity, bipartiteness, planarity, and colorability of Toeplitz graphs. Some Hamiltonian properties of undirected Toeplitz graphs have been investigated in [1] and [5], while the directed case was studied in [6–8]. In [9], S. Malik and T. Zamfirescu started the investigation of the Hamiltonian connectedness of directed Toeplitz graphs. For the indirected case, in [9] it is proven that $T_n\langle 1,2\rangle$ is Hamiltonian connected only for $n = 3$, while $T_n\langle 1,2,s\rangle$ is Hamiltonian connected for all values of n and s. It will become clear that, concerning k, the first relevant case is $k = 3$. In this paper, we are completing the picture of Hamiltonian connectedness of Toeplitz graphs, more precisely of $T_n\langle t_1,t_2\rangle$, $T_n\langle 1,3,s\rangle$ and $T_n\langle 1,4,s\rangle$.

Let T be a Toeplitz graph and p, q be two vertices of T, such that $p < q$. By $P_{p,q}$ we mean a path from p to $p+1$ containing all vertices in $\{p, p+1, p+2, \ldots, q-2, q-1, q\}$, and by $P_{q,p}$ we mean a path from q to $q-1$, containing the same vertices. The existence of $P_{p,q}$ or $P_{q,p}$ is not guaranteed.

We start with a few simple results.

Theorem 1. *For $n \neq 3$, $T_n\langle t_1,t_2\rangle$ is not Hamiltonian connected.*

Proof. Assume $T = T_n\langle t_1,t_2\rangle$ for $n \neq 3$ is Hamiltonian connected. Then there exists a Hamiltonian path from t_1+1 to t_2+1. But the path from t_1+1 to t_2+1 containing 1 is unique and is of length 2. This leads to a contradiction. Hence, T is not Hamiltonian connected. □

Theorem 2. *The Toeplitz graph $T_n\langle t_1,t_2,t_3,\ldots, t_k\rangle$ is not Hamiltonian connected if $t_1, t_2, t_3, \ldots, t_k$ are all odd.*

Proof. A bipartite graph is not Hamiltonian connected, and if $t_1, t_2, t_3, \ldots, t_k$ are all odd, then the graph $T_n\langle t_1,t_2,t_3,\ldots, t_k\rangle$ is bipartite. □

Corollary. *$T_n\langle 1,3,s\rangle$ is not Hamiltonian connected, when s is odd .*

Theorem 3. *If both n and t are odd, then $T_n\langle 1,t,n-1\rangle$ is not Hamiltonian connected.*

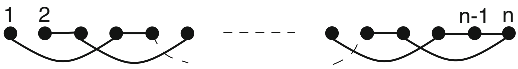
Proof. Let, for t odd, $T = T_n\langle 1,t,n-1\rangle$, where $n \geq t+2$ is an odd integer. Assume that T is Hamiltonian connected, then there exists a Hamiltonian path H between two even vertices x and y of T. The path H either contains the edge $(1,n)$ or not.

If H contains the edge $(1,n)$, we can contract it to a single vertex, because both vertices of the edge have the same parity (both are odd). After contraction, the resulting path H' is of even order, and the number of even vertices is equal to the number of odd vertices. But the end vertices of H' are even, which leads to a contradiction.

Next, we assume that H does not contain the edge $(1,n)$. But T without the edge $(1,n)$ becomes $T_n\langle 1,t\rangle$, which is a bipartite graph. Again, H cannot be a Hamiltonian path of T, and this completes the proof. □

Lemma 1. *If n is even, then $T_n\langle 1,3\rangle$ admits a Hamiltonian path from 1 to 2 and, by symmetry, another one from n to $n-1$.*

Fig. 1



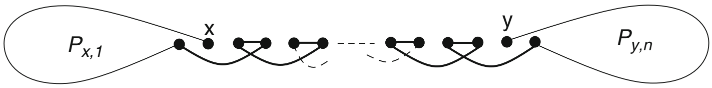


Fig. 2

Fig. 3

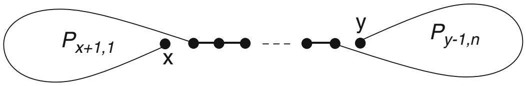
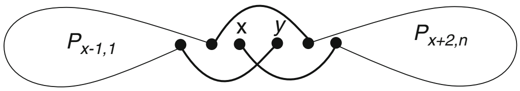


Fig. 4



Proof. See Fig. 1, for a Hamiltonian path in $T = T_n\langle 1, 3\rangle$, from vertex 1 to vertex 2, for even $n \geq 4$. A similar Hamiltonian path from vertex n to vertex $n - 1$ exists in T, due to the symmetry of Toeplitz graphs. This completes the proof. □

Lemma 2. *Let p, q be two distinct vertices of $T_n\langle 1, 3\rangle$. If $q - p$ is odd then paths $P_{p,q}$ and $P_{q,p}$ exist in $T_n\langle 1, 3\rangle$.*

Proof. Apply Lemma 1 to the subgraph of $T_n\langle 1, 3\rangle$ spanned by $p, p + 1, \ldots q$. □

Theorem 4. *$T_n\langle 1, 3, s\rangle$ is Hamiltonian connected for all $n \geq s + 2$, if s is an even integer.*

Proof. Let $T = T_n\langle 1, 3, s\rangle$ be the Toeplitz graph, where $n \geq s + 2$. Then, there exist paths $P_{p,q}$ and $P_{q,p}$ in T, whenever $q - p$ is odd for $p < q$, by Lemma 2. Now, by using such paths of T, we prove that for any two distinct vertices x and y of T, there exists a Hamiltonian path from x to y. Take $x < y$. We split our proof into two main cases:

Case 1. n is even.

The following four subcases arise:

(*i*) *x is even, y is odd.*
In this case, $P_{x,1}$ and $P_{y,n}$ exist in T, and, with the help of these two paths, we obtain a Hamiltonian path $\left(P_{x,1}, x + 2, x + 1, \ldots, y - 1, y - 2, P_{y,n}\right)$ in T from x to y; see Fig. 2.

(*ii*) *x is odd, y is even.*
If $y \neq x + 1$, then a possible Hamiltonian path of T from x to y, $\left(P_{x+1,1}, x + 2, \ldots, P_{y-1,n}\right)$, is shown in Fig. 3.

When $y = x + 1$, then for $x = 1$ or $x = n - 1$, we use the path of Lemma 1, and for other values of x, we consider the path $(x, P_{x+2,n}, P_{x-1,1}, y)$; see Fig. 4.

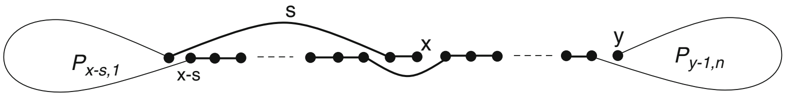


Fig. 5

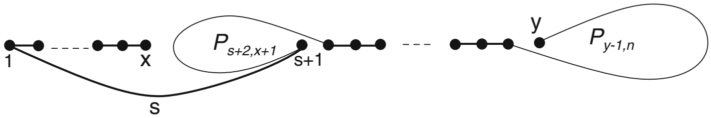


Fig. 6

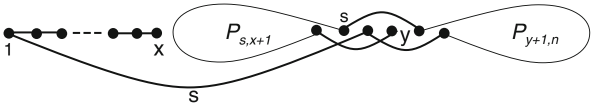


Fig. 7

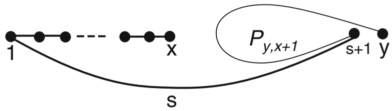


Fig. 8

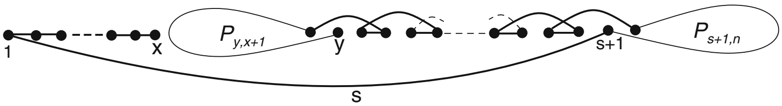


Fig. 9

(iii) *x and y are even.*

If $x > s$, then a Hamiltonian path from x to y is $(x,\ x-1,\ P_{x-s,1},\ x-s+1,\ \ldots,\ x-2,\ x+1,\ x+2,\ \ldots,\ y-2,\ P_{y-1,n})$ (see Fig. 5).

If $x \le s$, then we have four subcases to discuss:

(a) For $y > s+2$, we consider a Hamiltonian path $(x,\ x-1,\ x-2,\ \ldots,\ 2,\ 1,\ P_{s+2,x+1},\ s+3,\ \ldots,\ y-2,\ P_{y-1,n})$ between x and y; see Fig. 6.

(b) When $y = s+2 \ne n$, then a possible Hamiltonian path joining x and y is $(x,\ x-1,\ x-2,\ \ldots,\ 2,\ 1,\ s+1,\ P_{y+1,n}, P_{s,x+1},\ s+2 = y)$; see Fig. 7.

(c) If $y = s+2 = n$, then a Hamiltonian path from x to y is $(x,\ x-1,\ x-2,\ \ldots,\ 3,\ 2,\ 1, P_{n,x+1})$; see Fig. 8.

(d) Finally, for $y \le s$. A Hamiltonian path joining x and y is $(x,\ x-1,\ x-2,\ \ldots,\ 2,\ 1,\ P_{s+1,n},\ s-1,\ s,\ s-3,\ s-2,\ \ldots,\ y+1,\ y+2, P_{y,x+1})$ (see Fig. 9).

(iv) *x is odd, y is odd.*

This case is symmetric to case (iii). (Denote vertex i by $n+1-i$.)

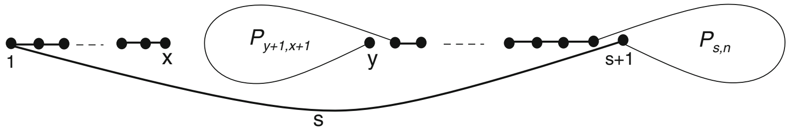


Fig. 10

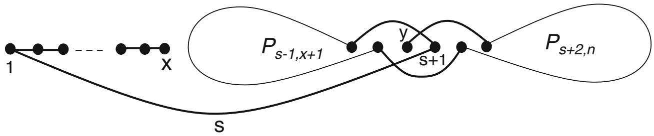


Fig. 11

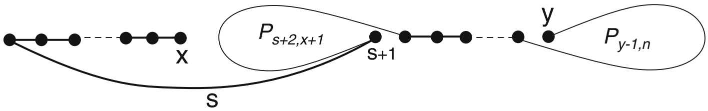


Fig. 12

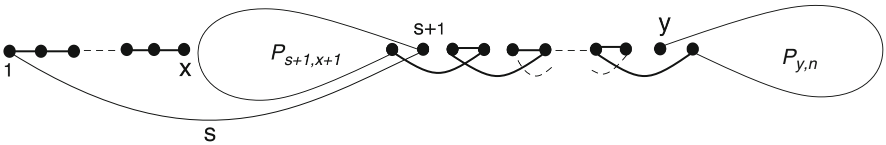


Fig. 13

Case 2. n is odd.

Again, we consider the following subcases:

(i) *x and y are of different parity.*
First, we assume that $y < s$. Then a Hamiltonian path joining x and y is $(x,\ x-1,\ x-2, \ldots, 2,\ 1,\ P_{s,n},\ s-1,\ \ldots,\ y+2, P_{y+1,x+1})$ (see Fig. 10).
If $y = s$, then a Hamiltonian path joining x and y is $(x,\ x-1,\ \ldots, 2,\ 1,\ s+1,\ P_{s-1,x+1},\ P_{s+2,n},\ y)$ (see Fig. 11).
Next suppose that $x \leq s$ and $y > s+1$.

- (a) If x is even, then a Hamiltonian path joining x and y is $(x,\ x-1,\ x-2, \ldots, 2,\ 1,\ P_{s+2,x+1},\ s+3,\ \ldots,\ y-2, P_{y,n})$; see Fig. 12.
- (b) If x is odd, then a Hamiltonian path $(x, x-1, \ldots, 1, P_{s+1,x+1}, s+3, \ldots, y-1, y-2, P_{y,n})$, joining x and y, is shown in Fig. 13.

When $2 < x < s$ and $y = s+1$, we consider a Hamiltonian path $(x, x-1, \ldots, 4, 1, 2, 3, P_{s+2,n}, P_{s+2,x+1})$ from x to y (see Fig. 14).

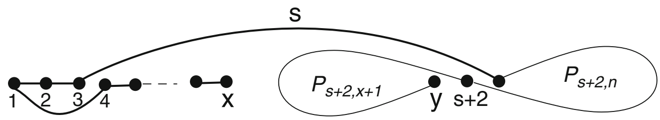


Fig. 14

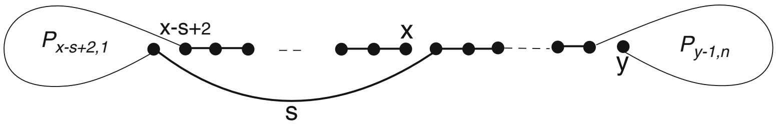


Fig. 15

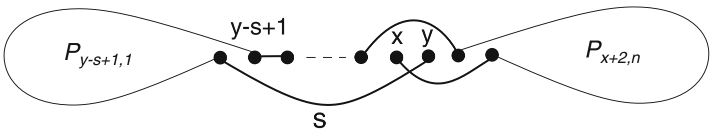


Fig. 16

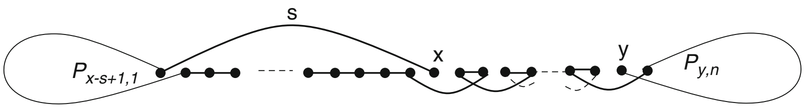


Fig. 17

If $x = 2$ and $y = s + 1$, then a Hamiltonian path joining x and y is $(x = 2, 1, P_{3,y-1}, P_{s+2,n}, y)$.
Finally, here we consider the case $x > s$.

(a) If x is even and $y \neq x + 1$, then a Hamiltonian path from x to y is $(x, x-1, \ldots, x-s+1, P_{x-s+2,1}, x+1, x+2, \ldots, y-2, P_{y-1,n})$; see Fig. 15.

If $y = x + 1$, then a Hamiltonian path from x to y is $(x, P_{x+2,n}, x-1, x-2, \ldots, y-s+2, P_{y-s+1,1}, y)$ (see Fig. 16).

(b) If x is odd, then a Hamiltonian path from x to y is $(x, P_{x-s+1,1}, x-s+2, \ldots, x-1, x+2, x+1, \ldots, y-1, y-2, P_{y,n})$; see Fig. 17.

(ii) *x and y are even.*
The following subcases arise:

If $x = 2$ and $y \geq s + 2$, we use $(2, 1, s+1, s, \ldots, 4, 3, s+3, s+2, s+5, \ldots, y-1, y-2, y+1, P_{y,n})$, the Hamiltonian path between x and y (see Fig. 18).

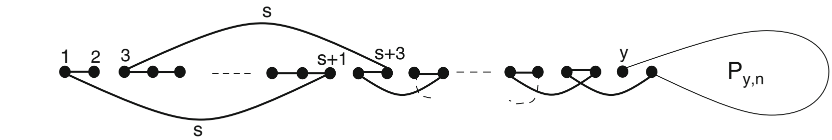


Fig. 18

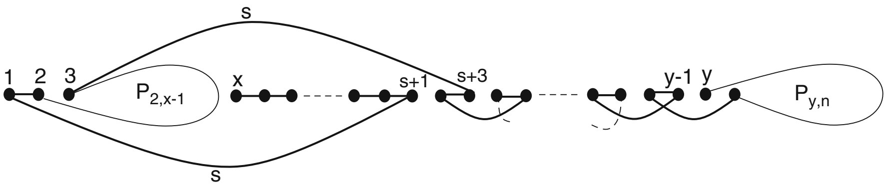


Fig. 19

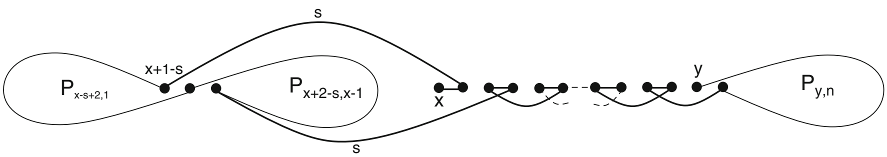


Fig. 20

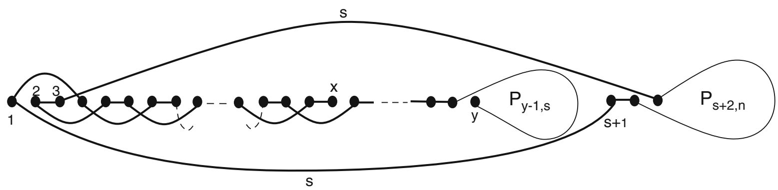


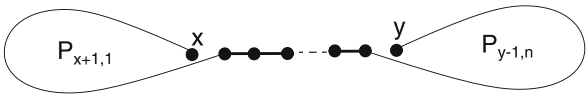
Fig. 21

When $4 \leq x \leq s$ and $y \geq s+2$, then a Hamiltonian path joining x and y is $(x, x+1, \dots, s+1, 1, P_{2,x-1}, s+3, s+2, s+5, \dots, y-1, y-2, P_{y,n})$, shown in Fig. 19.

If $x > s$, we have $(x, x+1, P_{x-s+2,1}, P_{x-s+2,x-1}, x+3, x+2, \dots, y-1, y-2, P_{y,n})$, the Hamiltonian path from x to y (see Fig. 20).

If $y < s$, the path from x to y as desired is $(x, x-1, x-4, x-5, \dots, 4, 1, s+1, P_{s+2,n}, 3, 2, 5, 6, \dots, x-3, x-2, x+1, x+2, \dots, y-2, P_{y-1,s})$, when $x \equiv 0 (\mathrm{mod}\, 4)$, and $(x, x-1, x-4, x-5, \dots, 6, 5, 2, 3, P_{s+2,n}, s+1, 1, 4, 7, 8, \dots, x-3, x-2, x+1, x+2, \dots, y-2, P_{y-1,s})$, when $x \equiv 2 (\mathrm{mod}\, 4)$; see also Fig. 21.

Fig. 22



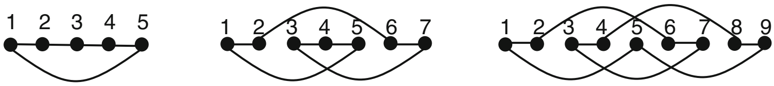


Fig. 23

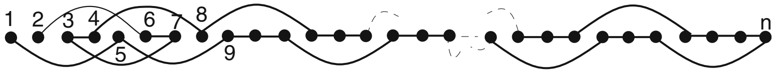


Fig. 24

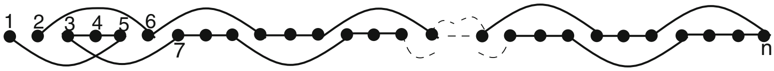


Fig. 25

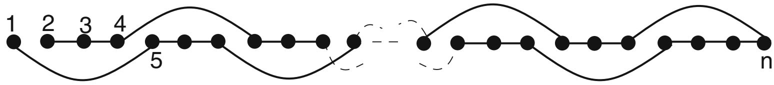


Fig. 26

(*iii*) *x and y are odd.*

In this simple case a Hamiltonian path from x to y is $(P_{x+1,1}, x+2, \ldots, y-3, y-2, P_{y-1,n})$ (see Fig. 22).

Now the proof is complete.

□

Lemma 3. *For $n = 5$ and all $n \geq 7$, $T_n\langle 1, 4\rangle$ admits a Hamiltonian path from* 1 *to* 2 *and, by symmetry, another one from n to $n-1$.*

Proof. $T_n\langle 1, 4\rangle$ is Hamiltonian for all values of n except 6. See Fig. 23 for a Hamiltonian cycle in $T_n\langle 1, 4\rangle$, when $n \in \{5, 7, 9\}$. These cycles are unique and we use them to find a Hamiltonian path from 1 to 2 in $T_n\langle 1, 4\rangle$.

For any $n \equiv 0 (\mathrm{mod}\, 3)$, a suitable path is obtained by using the Hamiltonian cycle in $T_9\langle 1, 4\rangle$; see Fig. 24.

To obtain such a path when $n \equiv 1 (\mathrm{mod}\, 3)$, we use the Hamiltonian cycle found in $T_7\langle 1, 4\rangle$; see Fig. 25.

For $n \equiv 2 (\mathrm{mod}\, 3)$, the cycle $T_5\langle 1, 4\rangle$ is employed; see Fig. 26.

Now, because of the symmetry of the Toeplitz graph, we also have a Hamiltonian path from n to $n-1$. □

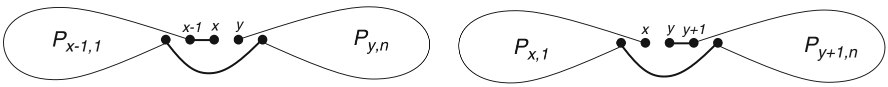


Fig. 27

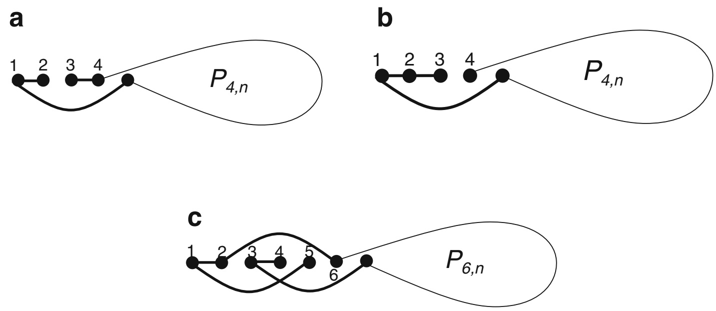


Fig. 28

Lemma 4. *Let p, q be two distinct vertices of $T_n\langle 1, 4\rangle$. If $q - p \neq 2, 3, 5$, then there exist paths $P_{p,q}$ and $P_{q,p}$ in $T_n\langle 1, 4\rangle$.*

Proof. See Lemma 3. □

Theorem 5. *$T_n\langle 1, 4, s\rangle$ is Hamiltonian connected for all s and $n \geq 15$.*

Proof. For $n \geq 15$, let x and y be distinct vertices of the Toeplitz graph $T = T_n\langle 1, 4, s\rangle$. Assume that $x < y$. To prove the result we show that there exists a Hamiltonian path between x and y.

Case 1. $y = x + 1$.

If $x = 1$ or $n - 1$, we have a desired path due to Lemma 3.

When $5 \leq x \leq n - 5$, then a Hamiltonian path between x and y is either $(x, P_{x-1,1}, P_{y,n})$ or $(P_{x,1}, P_{y+1,n}, y)$ (see Fig. 27).

When $2 \leq x \leq 4$, see Fig. 28 for a Hamiltonian path between x and y.

For $n - 4 \leq x \leq n - 2$, the desired Hamiltonian paths are symmetric to the paths for $x \in \{2, 3, 4\}$.

Case 2. If $y \neq x + 1$, the following three subcases arise:

(*i*) $x \leq n - 5$ and $y \in \{3, 4, \ldots, n - 5, n - 3\}$.
(*ii*) $x \leq n - 5$ and $y \in \{n - 4, n - 2, n - 1, n\}$.
(*iii*) $x \geq n - 4$.

Subcase (*i*). Let $y \in \{3, 4, \ldots, n - 5, n - 3\}$.

(*a*) First, we assume the case when $x \in \{4, 6, 7, \ldots, n-5\}$. Now $(P_{x+1,1}, x + 2, x + 3, \ldots, y - 2, P_{y-1,n})$ is a required Hamiltonian path between x and y (see Fig. 29).

(*b*) If $x = 1$, then a desired path between 1 and y is shown in Fig. 30.

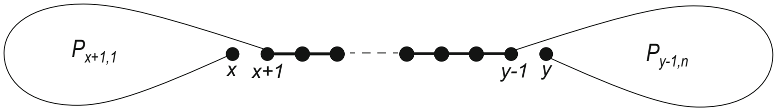
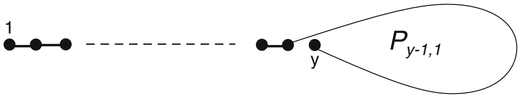


Fig. 29

Fig. 30



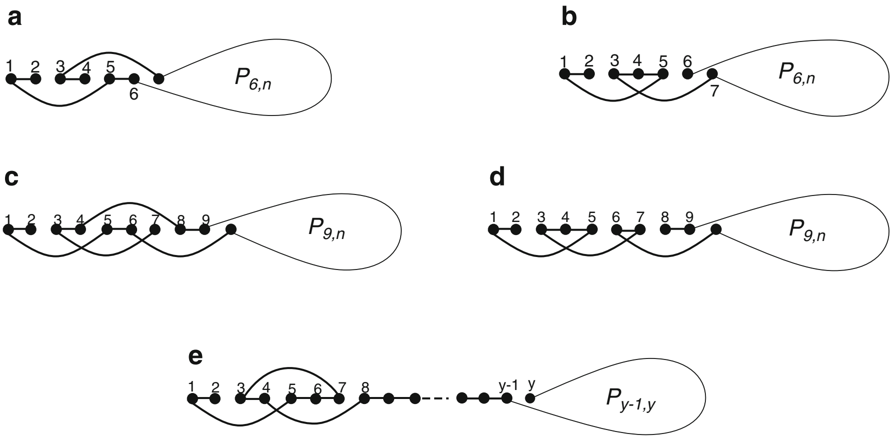


Fig. 31 (**a**) A Hamiltonian path between 2 and 4. (**b**) A Hamiltonian path between 2 and 6. (**c**) A Hamiltonian path between 2 and 7. (**d**) A Hamiltonian path between 2 and 8. (**e**) A Hamiltonian path between 2 and y, where $y \geq 9$

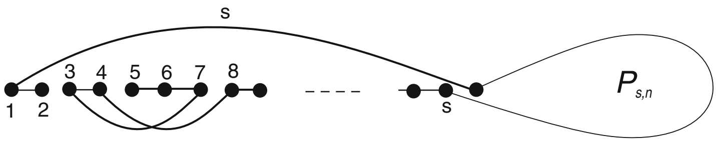


Fig. 32

(c) If $x = 2$ and $y \neq 5$, then Hamiltonian paths between 2 and different values of y are shown in Fig. 31.

When $x = 2$ and $y = 5$, to get a desired path, we use the difference s along with differences 1 and 4. See Fig. 32, for such a path when $s \in \{8, 9, 10, \ldots, n-6, n-4\}$.

When $s = 5, 6, 7$, see Fig. 33.

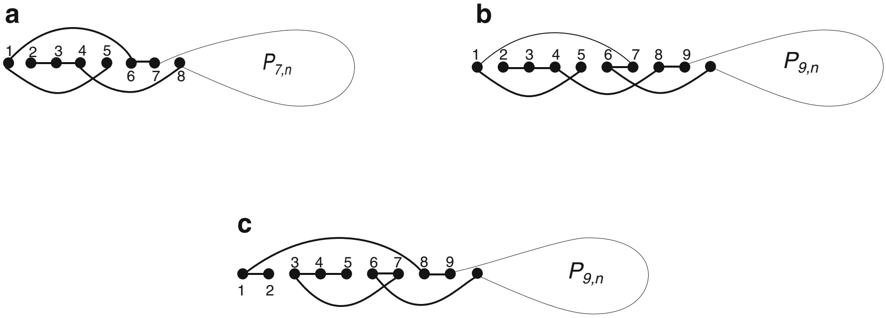


Fig. 33 (**a**) $s = 5$. (**b**) $s = 6$. (**c**) $s = 7$

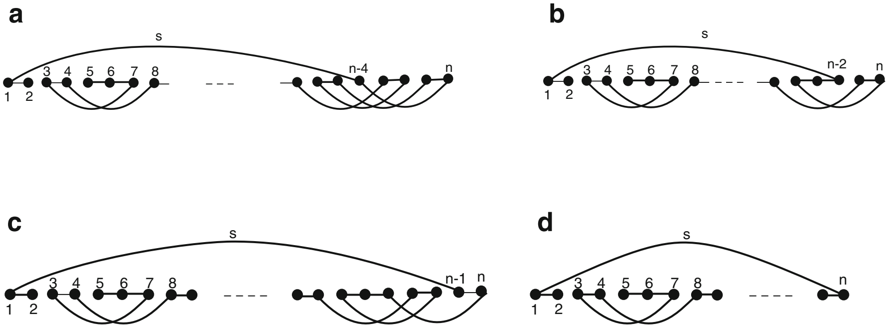


Fig. 34 (**a**) $s = n - 5$. (**b**) $s = n - 3$. (**c**) $s = n - 2$. (**d**) $s = n - 1$

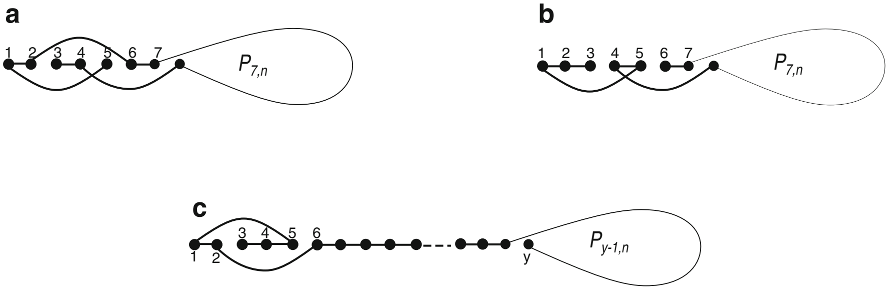


Fig. 35 (**a**) A Hamiltonian path between 3 and 5. (**b**) A Hamiltonian path between 3 and 6. (**c**) A Hamiltonian path between 3 and $y \geq 7$

And, for $s = n - 5, n - 3, n - 2, n - 1$, see Fig. 34

(d) If $x = 3$, then for a Hamiltonian path between 3 and y, see Fig. 35.

(e) If $x = 5$ and $y \neq 8$, a desired Hamiltonian path is shown in Fig. 36. When $y = 8$ and $n \neq 15, 17$, we use the path shown in Fig. 37. For $n = 15$ and $n = 17$, see Figs. 38 and 39, respectively.

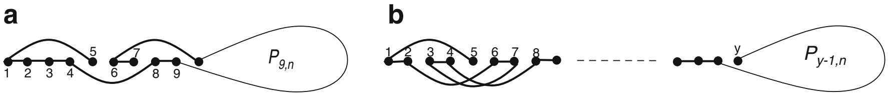


Fig. 36 (**a**) A Hamiltonian path between 5 and 7. (**b**) A Hamiltonian path between 5 and y, where $y \geq 9$

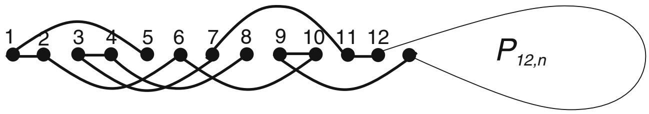


Fig. 37

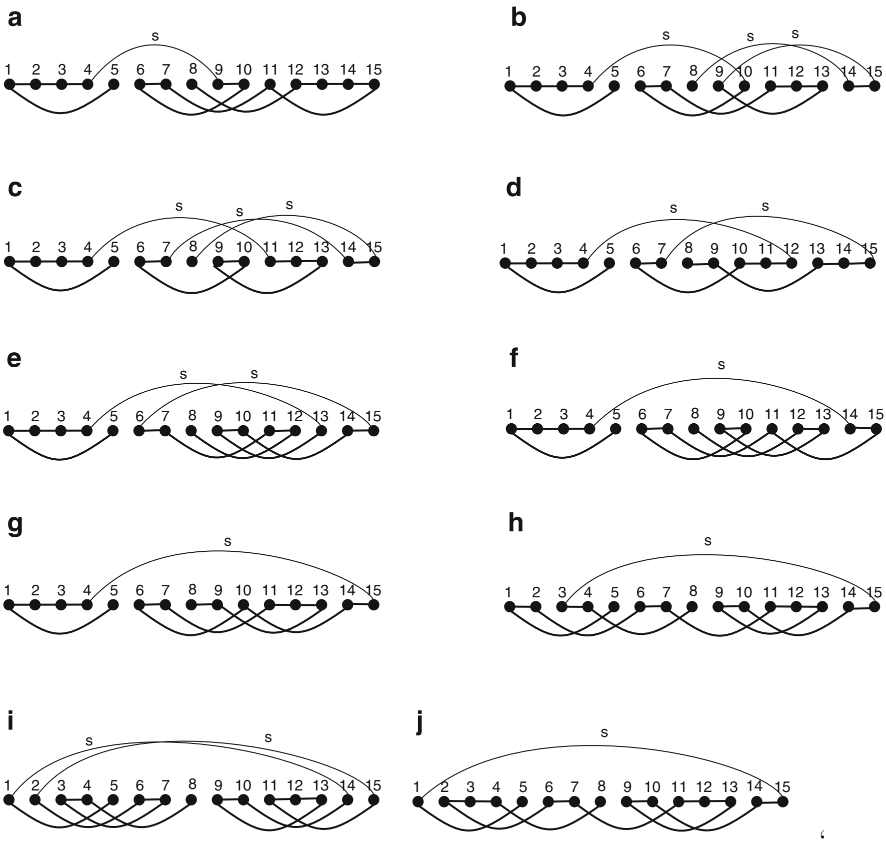


Fig. 38 Hamiltonian paths between 5 and 8 for different values of s, when $n = 15$. (**a**) $s = 5$. (**b**) $s = 6$. (**c**) $s = 7$. (**d**) $s = 8$. (**e**) $s = 9$. (**f**) $s = 10$. (**g**) $s = 11$. (**h**) $s = 12$. (**i**) $s = 13$. (**j**) $s = 14$

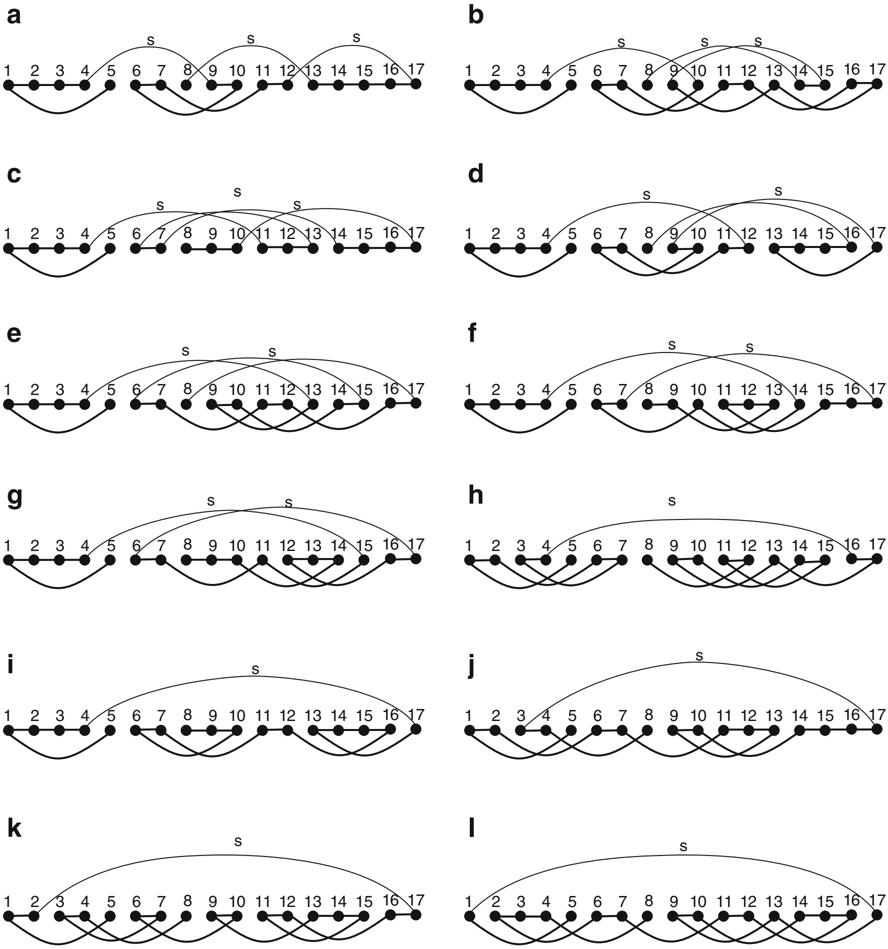


Fig. 39 Hamiltonian paths between 5 and 8 for different values of s, when $n = 17$. (**a**) $s = 5$. (**b**) $s = 6$. (**c**) $s = 7$. (**d**) $s = 8$. (**e**) $s = 9$. (**f**) $s = 10$. (**g**) $s = 11$. (**h**) $s = 12$. (**i**) $s = 13$. (**j**) $s = 14$. (**k**) $s = 15$. (**l**) $s = 16$

Subcase (*ii*). This subcase is symmetrical to $x \in \{1, 2, 3, 5\}$ and $y \geq 6$. It was treated inside of (i) except for the cases $y = n-4, n-2, n-1, n$.

To obtain a Hamiltonian path from $x \in \{1, 2, 3, 5\}$ to $y \in \{n-4, n-2, n-1, n\}$, we first collect the four Hamiltonian paths in $T_8\langle 1, 4\rangle$ from $x \in \{1, 2, 3, 5\}$ to 8; see Fig. 40. Symmetrically, we have paths in $T_n\langle 1, 4\rangle$ from $y \in \{n-4, n-2, n-1, n\}$ to $n-7$, of vertex set $\{n-7, n-6, \ldots, n\}$. Joining 8 to $n-7$ by the direct path (8,9,...,$n-7$) gives the desired Hamiltonian path in $T_n\langle 1, 4\rangle$ from x to y.

Subcase (*iii*). This subcase is symmetrical with $y \leq 5$, treated inside of (i).

□

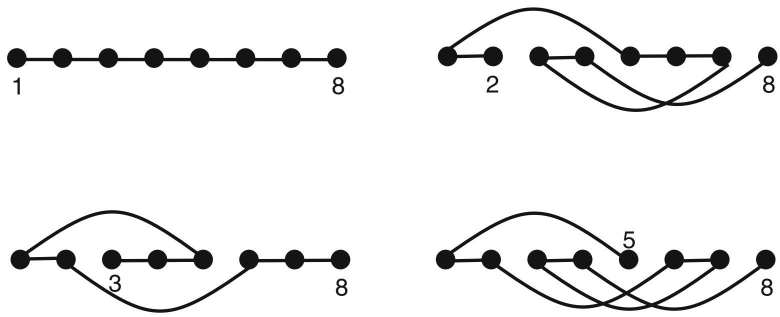


Fig. 40

To see whether $T_n\langle 1, 4, s\rangle$ is Hamiltonian connected or not, for $6 \leq n \leq 14$, see the following table:

	Hamiltonian connected when s is
$T_6\langle 1, 4, s\rangle$	
$T_7\langle 1, 4, s\rangle$	
$T_8\langle 1, 4, s\rangle$	5, 7
$T_9\langle 1, 4, s\rangle$	5, 8
$T_{10}\langle 1, 4, s\rangle$	5, 6, 7, 9
$T_{11}\langle 1, 4, s\rangle$	5, 7, 8, 10
$T_{12}\langle 1, 4, s\rangle$	5, 6, 7, 8, 9, 11
$T_{13}\langle 1, 4, s\rangle$	for all s
$T_{14}\langle 1, 4, s\rangle$	5, 6, 7, 8, 9, 10, 11, 13

Missing values for s mean that the corresponding Toeplitz graph is not Hamiltonian connected. This was verified by using a computer.

References

1. van Dal, R., Tijssen, G., Tuza, Z., van der Veen, J.A.A., Zamfirescu, Ch., Zamfirescu, T.: Hamiltonian properties of Toeplitz graphs. Discret. Math. **159**, 69–81 (1996)
2. Euler, R.: Characterizing bipartite Toeplitz graphs. Theor. Comput. Sci. **263**, 47–58 (2001)
3. Euler, R., LeVerge, H., Zamfirescu, T.: A characterization of infinite, bipartite Toeplitz graphs. In: Tung-Hsin, K. (ed.) Combinatorics and Graph Theory 95, Vol. 1. Academia Sinica, pp. 119–130. World Scientific, Singapore (1995)
4. Euler, R., Zamfirescu, T.: On planar Toeplitz graphs. Graphs Comb. **29**, 1311–1327 (2013)
5. Heuberger, C.: On Hamiltonian Toeplitz graphs. Discret. Math. **245**, 107–125 (2002)
6. Malik, S.: Hamiltonian cycles in directed Toeplitz graphs II. Ars Comb. (to appear)

7. Malik, S.: Hamiltonicity in directed Toeplitz graphs of maximum (out or in) degree 4. Util. Math. **89**, 33–68 (2012)
8. Malik, S., Qureshi, A.M.: Hamiltonian cycles in directed Toeplitz graphs. Ars Comb. **109**, 511–526 (2013)
9. Malik, S., Zamfirescu, T.: Hamiltonian connectedness in directed Toeplitz graphs. Bull. Math. Soc. Sci. Math. Roum. **53**(101) No. 2, 145–156 (2010)