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When is a Disk Trapped by Four Lines?

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Abstract We use homological and geometrical methods to study the problem of determining when a convex disk is trapped by four lines.

Keywords Convex disk · Trapped

1 Introduction

The problem of holding a convex body with a circle has been studied extensively by several authors; see [1,2,6]. They studied the problem of determining when a convex

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body is trapped by a fixed circle. We are interested in this paper in the similar problem of determining when a convex disk is trapped by four lines. Here, we use the term *convex disk* with the meaning—used by the old Hungarian school—of a planar convex body.

Let Ω be a finite collection of lines in Euclidean 3-space $\mathbb{R}^3$ and let $D \subset \mathbb{R}^3$ be a convex disk with the property that the relative interior of D does not intersect any line of Ω transversally. We say that D is trapped by Ω if D cannot be moved continuously through infinity without the relative interior of D intersecting transversally a line of Ω. For example, six lines determined by the 1-skeleton of a tetrahedron of side $\frac{\sqrt{3}}{2}$ always trap a circular disk of diameter slightly smaller than or equal to 1, but it is not difficult to see that the union of three lines in $\mathbb{R}^3$ does not trap a circular disk. The purpose of this paper is to study the problem of determining when a convex disk is trapped by four lines.

In Sect. 2, we use homology theory to find criteria under which a convex disk is trapped by four lines. This enables us to prove, in Sect. 3, criteria in terms of the girth map. The girth map measures the girth of a collection of lines with respect to a given position of our convex disk. Section 4 is devoted to the notion of immobilization, and shows that a strictly convex disk is trapped by four lines provided they immobilize a quadrangle or a triangle.

Finally, we show in the last section that every convex disk with a C^2 boundary can be trapped by 4 lines.

2 Homological Characterization of the Case When a Convex Disk is Trapped by Four Lines

Let $SE(3)$ be the special Euclidean group of rigid motions of $\mathbb{R}^3$ with subgroups $T(3)$, the group of translations, homeomorphic to $\mathbb{R}^3$, and $SO(3)$ of rotations in $\mathbb{R}^3$, homeomorphic to $\mathbb{RP}^3$. In fact, $SE(3)$ is the semi direct product of $SO(3)$ with $T(3)$, but topologically $SE(3)$ is homeomorphic to $\mathbb{R}^3 \times SO(3)$.

From here on, let F be a convex disk in $\mathbb{R}^2 \subset \mathbb{R}^3$ that contains the origin in its interior and let Γ be the subgroup of $SE(3)$ consisting of all rigid motions that keep F fixed; that is, Γ is the group of isometries of F. Then $SE(3)/\Gamma$ represent all possible positions of the convex disk F in $\mathbb{R}^3$.

Let $\Omega = L_1 \cup L_2 \cup L_3 \cup L_4$ be a collection of four lines in $\mathbb{R}^3$. Let $\mathbb{D}_i$ be the collection of disks of the shape of F, contained in $\mathbb{R}^3$, whose relative interior intersects L_i transversally. That is,

$$\mathbb{D}_i = \{\gamma\Gamma \in SE(3)/\Gamma \mid \text{ the relative interior of } \gamma F \text{ intersects } L_i \text{ transversally}\}.$$

Note that each $\mathbb{D}_i$ is open in $SE(3)/\Gamma$. Let $\mathbb{D}(\Omega) = \mathbb{D}_1 \cup \mathbb{D}_2 \cup \mathbb{D}_3 \cup \mathbb{D}_4$. We are interested in the compact components of $SE(3)/\Gamma - \mathbb{D}(\Omega)$; that is, a copy of the convex disk F is trapped by the lines $\Omega = \{L_1, L_2, L_3, L_4\}$ if and only if $SE(3)/\Gamma - \mathbb{D}(\Omega)$ has a compact component.

Now let $p\colon X \to X$ be a fiber bundle with compact fiber and let U be an open subset of X. It is easy to see that $X - U$ has a compact component if and only if $X - p^{-1}(U)$

has a compact component. So $SE(3)/\Gamma - \mathbb{D}(\Omega)$ has a compact component if and only if $SE(3) - D(\Omega)$ has a compact component, where now

$$D_i = \{\gamma \in SE(3) \mid \text{the relative interior of } \gamma F \text{ intersects } L_i \text{ transversally}\},$$

and $D(\Omega) = D_1 \cup D_2 \cup D_3 \cup D_4$. Note that every D_i is an open subset of $\mathbb{R}^3 \times SO(3)$ and hence $D(\Omega)$ is also an open subset of $\mathbb{R}^3 \times SO(3)$. Then by Lemma 1 below, a copy of the convex disk F is trapped by the lines of $\Omega = \{L_1, L_2, L_3, L_4\}$ if and only if

$$H_5(D(\Omega)) \neq 0.$$

Similarly, a copy of the convex disk F is trapped by the lines $\{L_1, L_2, L_3\}$ if and only if

$$H_5(D_1 \cup D_2 \cup D_3) \neq 0.$$

In this paper we always use Cech homology groups with $\mathbb{Z}_2$ coefficients.

Lemma 1 *Let $U \subset \mathbb{R}^3 \times SO(3)$ be an open set. Then the complement of U has a compact component if and only if $H_5(U) \neq 0$.*

Proof First note that $\mathbb{R}^3 \times S^3$ is a double covering of $\mathbb{R}^3 \times SO(3)$, where S^3 denotes the unit sphere of $\mathbb{R}^4$. For a double covering map $\pi : X \to Y$, Y is compact ($H_5(Y) = 0$) if and only if X is compact (respectively $H_5(X) = 0$). Hence, we shall prove our lemma replacing $SO(3)$ by S^3. The strategy of the proof is to use the fact that, for an open subset $U \subset \mathbb{R}^n$, the complement of U in $\mathbb{R}^n$ has a bounded component if and only if $H_{n-1}(U) \neq 0$. We shall consider $\mathbb{R}^3 \times S^3$ to be an open subset of $\mathbb{R}^6$. In fact, $\mathbb{R}^3 \times S^3$ is homeomorphic to $\mathbb{R}^6 - L$, where L is a two-dimensional affine plane. This is so because $\mathbb{R}^3 \times S^3 = \mathbb{R}^2 \times (\mathbb{R} \times S^3)$ homeomorphic to $\mathbb{R}^2 \times (\mathbb{R}^4 - \{0\}) \subset \mathbb{R}^6$. Now let $U \subset \mathbb{R}^6 - L$ be an open set. Then $(\mathbb{R}^n - L) - U$ has a compact component if and only if $\mathbb{R}^n - U$ has a compact component if and only if $H_5(U) \neq 0$. □

The following technical lemma will be useful.

Lemma 2 *Let L_1, L_2, L_3, L_4 be four lines in $\mathbb{R}^3$ and let F be a convex disk in $\mathbb{R}^2 \subset \mathbb{R}^3$ that contains the origin in its interior. If $D_i = \{\gamma \in SE(3) \mid$ the relative interior of γF intersects L_i transversally$\}$, $i = 1, 2, 3, 4$, then*

1. *For $k = 1, 2, 3, 4$, let $\pi : \bigcap_1^k D_i \to \pi(\bigcap_1^k D_i)$ be the restriction of the projection map $\Pi : \mathbb{R}^3 \times SO(3) \to SO(3)$.*
   Then π is a homotopy equivalence.
2. *$H_j(D_i) = 0$, for $j \geq 4$, $i = 1, 2, 3, 4$.*
3. *$H_j(\bigcap_1^k D_i) = 0$, for $j \geq 3$ and $k = 2, 3, 4$.*
4. *$H_j(D_1 \cup D_2) = 0$, for $j \geq 4$.*

5. $H_j(D_1 \cap (D_2 \cup D_3)) = 0$, *for* $j \geq 4$.
6. $H_5(D_1 \cup D_2 \cup D_3 \cup D_4) = H_4([D_1 \cup D_2] \cap [D_3 \cup D_4])$.

Proof We want to prove that π is a homotopy equivalence by proving that the fibers $\pi^{-1}(\gamma)$ are contractible. Let $\gamma_0 \in \Pi(D_i) \subset SO(3)$. We will consider $\Pi^{-1}(\gamma_0) \cap D_i = \{x \in \mathbb{R}^3 \mid$ the relative interior of $x + \gamma_0 F$ intersects L_i transversely$\}$ = relint$(\gamma_0 F) \times L_i$, which is convex and hence contractible. For $k = 1, 2, 3, 4$, let us consider $\pi^{-1}(\gamma_0)$, where $\pi : \bigcap_1^k D_i \to \pi(\bigcap_1^k D_i)$ is the restriction of the projective map $\Pi : \mathbb{R}^3 \times SO(3) \to SO(3)$. Therefore $\pi^{-1}(\gamma_0) = \Pi^{-1}(\gamma_0) \cap \bigcap_1^k D_i = \bigcap_1^k (\Pi^{-1}(\gamma_0) \cap D_i)$, which is the intersection of k convex cylinders and hence it is contractible.

Now (2) follows from the fact that $\pi(D_i) = SO(3)$, and therefore D_i has the homotopy type of $SO(3)$. Similarly, (3) follows from the fact that for $k = 2, 3, 4$, $\pi(\bigcap_1^k D_i)$ is properly contained in $SO(3)$. The Mayer–Vietoris sequence of the pair (D_1, D_2) together with (2) and (3) implies (4). Now (5) follows from the Mayer–Vietoris sequence of the pair $((D_1 \cap D_2); (D_1 \cap D_3))$ and (3). Finally, (6) follows from the Mayer–Vietoris sequence of the pair $((D_1 \cup D_2); (D_3 \cup D_4)$ and 4). □

In the case in which we have only three lines, a copy of the convex disk F is trapped by $\{L_1, L_2, L_3\}$ if and only if

$$H_5(D_1 \cup D_2 \cup D_3) \neq 0.$$

By the Mayer–Vietoris exact sequence of the pair $(D_1, D_2 \cup D_3)$, and Lemma 2(3, 4)

$$H_5(D_1 \cup D_2 \cup D_3) = H_4(D_1 \cap (D_2 \cup D_3)) = 0.$$

Thus three lines never trap a convex disk. The same conclusion can also be obtained using only geometric arguments.

Let us consider the following commutative diagram where the first row corresponds to the Mayer–Vietoris sequence of the pair $(D_1 \cap [D_3 \cup D_4]; D_2 \cap [D_3 \cup D_4])$, the second row corresponds to the Mayer–Vietoris sequence of the pair $(D_1 \cap D_3 \cap D_4; D_2 \cap D_3 \cap D_4)$, the third row corresponds to the Mayer–Vietoris sequence of the pair $(D_1 \cap D_3; D_2 \cap D_3)$ and the fourth row corresponds to the Mayer–Vietoris sequence of the pair $(D_1 \cap D_4; D_2 \cap D_4)$.

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & H_4([(D_1 \cup D_2)] \cap [D_3 \cup D_4]) & \overset{\partial_2}{\to} & H_3([D_1 \cap D_2] \cap [D_3 \cup D_4]) & \overset{\rho_*}{\to} & \begin{array}{c} H_3(D_1 \cap (D_3 \cup D_4)) \\ \oplus \\ H_3(D_2 \cap (D_3 \cup D_4)) \end{array} \to \\
 & & \partial_1 \downarrow & & \partial_4 \downarrow & & \partial_7 \oplus \partial_8 \downarrow \\
0 & \to & H_3([D_1 \cup D_2] \cap [D_3 \cap D_4]) & \overset{\partial_3}{\to} & H_2(D_1 \cap D_2 \cap D_3 \cap D_4) & \overset{i_*}{\to} & \begin{array}{c} H_2(D_1 \cap D_3 \cap D_4) \\ \oplus \\ H_2(D_2 \cap D_3 \cap D_4) \end{array} \to \\
 & & k_* \downarrow & & j_* \downarrow & & \downarrow \\
\begin{array}{c} 0 \\ \oplus \\ 0 \end{array} & \begin{array}{c} \to \\ \oplus \\ \to \end{array} & \begin{array}{c} H_2((D_1 \cup D_2) \cap D_3) \\ \oplus \\ H_2((D_1 \cup D_2) \cap D_4) \end{array} & \begin{array}{c} \partial_5 \\ \to \\ \oplus \\ \to \\ \partial_6 \end{array} & \begin{array}{c} H_2(D_1 \cap D_2 \cap D_3) \\ \oplus \\ H_2(D_1 \cap D_2 \cap D_4) \end{array} & \to & \\
 & & \downarrow & & \downarrow & &
\end{array}
$$

Similarly, the first column corresponds to the Mayer–Vietoris sequence of the pair $([D_1 \cup D_2] \cap D_3; [D_1 \cup D_2] \cap D_4)$, the second column corresponds to the Mayer–Vietoris sequence of the pair $(D_1 \cap D_2 \cap D_3; D_1 \cap D_2 \cap D_4)$, and the third column corresponds to the Mayer–Vietoris sequences of the pairs $(D_1 \cap D_3; D_1 \cap D_4)$ and $(D_2 \cap D_3; D_2 \cap D_4)$.

Lemma 3 *There is a copy of the convex disk F trapped by the lines L_1, L_2, L_3, L_4 if and only if in the commutative diagram above $H_4([D_1 \cup D_2] \cap [D_3 \cup D_4]) \neq 0$ precisely when there is $0 \neq \alpha \in H_2(D_1 \cap D_2 \cap D_3 \cap D_4)$ such that $i_*(\alpha) = 0 = j_*(\alpha)$.*

Proof We know that there is a copy of the convex disk F trapped by the lines L_1, L_2, L_3 and L_4 if and only if $H_5(D_1 \cup D_2 \cup D_3 \cup D_4) \neq 0$ if and only if $H_4([D_1 \cup D_2] \cap [D_3 \cup D_4]\} \neq 0$. Let $0 \neq x \in H_4([D_1 \cup D_2] \cap [D_3 \cup D_4])$. By the exactness of the diagram above, ∂_1 and ∂_2 are monomorphic, which implies that $\partial_1(x) \neq 0$ and $\partial_2(x) \neq 0$. Again, by the exactness of the diagram above, ∂_3 and ∂_4 are monomorphic, which implies that $0 \neq \alpha = \partial_3 \partial_1(x) = \partial_4 \partial_2(x)$. Furthermore, by exactness $i_*(\alpha) = i_* \partial_3 \partial_1(x) = 0$ and $j_*(\alpha) = j_* \partial_4 \partial_2(x) = 0$. Conversely, suppose that there is $0 \neq \alpha \in H_2(D_1 \cap D_2 \cap D_3 \cap D_4)$ such that $i_*(\alpha) = 0 = j_*(\alpha)$. By exactness, there is $\beta \in H_3([D_1 \cup D_2] \cap [D_3 \cap D_4])$ and $\gamma \in H_3([D_1 \cap D_2] \cap [D_3 \cup D_4])$ such that

$\partial_3(\beta) = \partial_4(\gamma) = \alpha$. So, $(\partial_5 \oplus \partial_6)k_*(\beta) = 0$ and similarly $(\partial_7 \oplus \partial_8)\rho_*(\gamma) = 0$. Hence, again by exactness in the third and fourth row, $k_*(\beta) = 0$ and by exactness in the third column $\rho_*(\gamma) = 0$. Finally, this implies that there is $x \in H_4([D_1 \cup D_2] \cap [D_3 \cup D_4])$ such that $\partial_1(x) = \beta$. □

The following useful lemma follows immediately from Lemma 3 and the fact that π is a homotopy equivalence. First some notation. Let $O = D_1 \cap D_2 \cap D_3 \cap D_4$, $O_1 = D_2 \cap D_3 \cap D_4$, $O_2 = D_1 \cap D_3 \cap D_4$, $O_3 = D_1 \cap D_2 \cap D_4$, $O_4 = D_1 \cap D_1 \cap D_3$.

Lemma 4 *There is a copy of the convex disk F trapped by the lines L_1, L_2, L_3, L_4 if and only if there is $0 \neq \alpha \in H_2(\pi(O))$ such that for $j = 1, 2, 3, 4$, $i^j_*(\alpha) = 0 \in H_2(\pi(O_j))$, where $i^j : \pi(O) \to \pi(O_j)$ denotes the inclusion.*

3 The Girth Map Ψ

Let Υ be a finite collection of lines and let $W = \{\gamma \in SO(3) \mid \gamma(\mathbb{R}^2)$ is an affine plane parallel to a pair of non intersecting lines of $\Upsilon\}$. Thus W consists of a finite set of pairwise disjoint curves $\{C_1, \ldots, C_\lambda\}$ in SO^3. Let $SO^3_\Upsilon = SO(3) - W$. The girth map

$$\Psi_\Upsilon : SO^3_\Upsilon \to (0, \infty)$$

is the continuous map defined as follows: for every $\gamma \in SO^3_\Upsilon$, let $\Psi_\Upsilon(\gamma)$ be the smallest positive real number t such that a translated copy of $t\gamma F$ intersects all lines of Υ. Note that the girth map Ψ_Υ is continuous because $\Psi^{-1}_\Upsilon((0, h))$ is open in SO^3_Υ and $\Psi^{-1}_\Upsilon((0, h])$ is closed in SO^3_Υ. Note also that $\Psi_\Upsilon(\gamma)$ tends to infinity when γ tends to one of the curves C_i. In our setting, for brevity let us denote Ψ_Ω by Ψ, where $\Omega = \{L_1, L_2, L_3, L_4\}$; and $\Psi_{\Omega - \{L_j\}}$ by Ψ_j , $i = 1, 2, 3, 4$. Note that $\pi(O) = \Psi^{-1}((0, 1))$.

The purpose of the following theorems is to characterize a convex disk trapped by the lines $\{L_1, L_2, L_3, L_4\}$. We first need a definition.

Let $\Lambda \subset SO^3_\Omega$ be a compact set. We say that Ψ has a *local maximum* h_0 at Λ if there is an open neighborhood U of Λ in SO^3_Ω such that $\Psi(\gamma) = h_0$ for every $\gamma \in \Lambda$, and $\Psi(\gamma) < h_0$ for every $\gamma \in U - \Lambda$.

Theorem 1 *Suppose there is a copy of the convex disk F trapped by the lines of $\Omega = \{L_1, L_2, L_3, L_4\}$. Then there is a local maximum $h_0 \geq 1$ of the girth map Ψ at $\Lambda \subset SO^3_\Omega$ such that for $j = 1, 2, 3, 4$, $\Psi_j(\gamma) < 1$ for every $\gamma \in \Lambda$.*

Proof An open set $U \subset SO^3_\Omega$ has a compact component if and only if there is $0 \neq \alpha \in H_2(U)$ such that for $i_*(\alpha) = 0 \in H_2(SO^3_\Omega)$, where $i^j : U \to SO^3_\Omega$ denotes the inclusion.

Suppose there is a copy of the convex disk F trapped by the lines L_1, L_2, L_3 and L_4, so that by Lemma 4, there is $0 \neq \alpha \in H_2(\pi(O))$ such that α is zero in $H_2(\pi(O_j))$, and hence is zero in $H_2(SO^3_\Omega)$, because $\pi(O_j) \subset SO^3_\Omega$. Then $SO^3_\Omega - \pi(O)$ has a compact component. We know that $\pi(O) = \Psi^{-1}((0, 1))$, so $SO^3_\Omega - \Psi^{-1}((0, 1))$ has a compact component $T \subset \pi(O_j)$, for $j = 1, 2, 3, 4$, which implies that Ψ has a local maximum $h_0 \geq 1$ at $\Lambda \subset T$ and for $j = 1, 2, 3, 4$, $\Psi_j(\gamma) < 1$ for every $\gamma \in \Lambda$. □

Concerning Theorem 1, note that the local maximality of Ψ may be attended at several positions. The second technical condition means that the local maximality is really a property of the 4 lines and not of 3 of them.

Theorem 2 *Suppose there is a local maximum h_0 of the map Ψ at $\Lambda \subset SO^3_\Omega$ such that for $j = 1, 2, 3, 4$, $\Psi_j(\gamma) < h_0$ for every $\gamma \in \Lambda$. Then there exists $\epsilon > 0$ such that for every $h_1 \in (h_0 - \epsilon, h_0]$ there is a copy of the convex disk $h_1 F$ trapped by the lines of $\Omega = \{L_1, L_2, L_3, L_4\}$.*

Proof Since Ψ has a local maximum h_0 at Λ, there is an open neighborhood U of Λ in SO^3_Ω such that $\Psi(\gamma) = h_0$ for every $\gamma \in \Lambda$, and $\Psi(\gamma) < h_0$ for every $\gamma \in U - \Lambda$. In fact, we may choose U and $\epsilon > 0$ in such a way that $h_0 - \epsilon < \Psi(\gamma) \leq h_0$ for every $\gamma \in U$. On the other hand, since $\Psi_j(\gamma) < h_0$ for every $\gamma \in \Lambda$, $\epsilon > 0$ is also chosen in such a way that $\Psi(\gamma) < h_0 - \epsilon$, for every $\gamma \in U$, $j = 1, 2, 3, 4$. This means that for every $\gamma \in U$ and $j = 1, 2, 3, 4$, the lines of $\Omega - L_j$ transversally intersect the relative interior of a translated copy of $(h_0 - \epsilon)\gamma F$.

Let $h_1 \in (h_0 - \epsilon, h_0]$. We use Lemma 4, but now with $h_1 F$ playing the role of F. This time $\pi(O) = \Psi^{-1}((0, h_1))$ and $U \subset \pi(O_j)$ for $j = 1, 2, 3, 4$. Then $SO^3_\Omega - \Psi^{-1}((0, h_1))$ has a compact component $T' \subset U \subset \pi(O_j)$, which implies the existence of $0 \neq \alpha \in H_2(\pi(0))$ which is zero in $H_2(\pi(O_j))$ for $j = 1, 2, 3, 4$. □

4 Immobilization and Imprisonment

First we give two important definitions.

Let Ω be a finite collection of lines in $\mathbb{R}^3$ and let $D \subset \mathbb{R}^3$ be a convex disk with the property that the relative interior of D does not intersect any line of Ω transversally. We say that Ω *immobilizes* D if any small rigid movement of D causes a line of Ω to penetrate the interior of D transversally.

For immobilization of convex bodies, see [3,4].

Remark If a convex disk is immobilized by the lines of Ω then it is also trapped, In fact, this case corresponds to the case in Theorems 1 and 2 in which the girth function has an isolated local maximum.

Let again Ω be a finite collection of lines in $\mathbb{R}^3$ and let $D \subset \mathbb{R}^3$ be a convex disk with the property that the relative interior of D does not intersect any line of Ω transversally and every line of Ω intersects D. We say that Ω *imprisons* D if after any small rigid movement of D, D is still intersecting all the lines of Ω.

While it is true that if a convex disk is immobilized by the lines of Ω then it is also trapped, the same is not necessarily true for imprisonment. That is, a convex disk may by imprisoned by Ω but not trapped.

Let $\gamma_0 F$ be a copy of the convex disk F trapped by the lines of $\Omega = \{L_1, L_2, L_3, L_4\}$, where $\gamma_0 \in SE(3)$, and let $h_1 \delta F$ be a similar copy of F, where $h_1 \geq 1$ and $\delta \in SE(3)$. We write $\gamma_0 F \sim h_1 \delta F$ if there are continuous functions

$h\colon [0,1] \to [1,\infty)$ and $\gamma\colon [0,1] \to SE(3)$ such that for every $t \in [0,1]$, the relative interior of the convex disk $h(t)\gamma(t)F$ does not intersect a line of Ω transversally, and furthermore $h(0) = 1$, $\gamma(0) = \gamma_0$, $h(1) = h_1$ and $\gamma(1) = \gamma$. The fact that $\gamma_0 F$ is trapped by the lines of Ω implies that $\gamma(t)F$ is also trapped by the lines of Ω, and hence that $h(t)$ is bounded; otherwise, if for some $t_0 \in [0,1]$, $h(t_0)$ is too big, then $\gamma(t_0)F$ can escape from Ω inside $\gamma(t)\mathbb{R}^2$. Let

$$h_1 = \text{Max}\{h \geq 1 \mid \gamma_0 F \sim h\delta F\}.$$

Lemma 5 *Suppose the convex disk $\gamma_o F$ is trapped by the lines of $\Omega = \{L_1, L_2, L_3, L_4\}$. Then for every $\gamma_0 F \sim h_1\delta F$, the convex disk $h_1\delta F$ is imprisoned by the lines of $\Omega = \{L_1, L_2, L_3, L_4\}$.*

Proof We start by proving that for every disk $h_1\delta F$ such that $\gamma_0 F \sim h_1\delta F$, we have that the lines of Ω intersect $h_1\delta F$. Suppose they do not. Since $h_1\delta F$ is trapped by Ω, then at least three of the lines, say $\{L_1, L_2, L_3\}$, intersect $h_1\delta F$, but $L_4 \cap h_1\delta F = \phi$. Since a convex disk is never trapped by three lines, the disk $h_1\delta F$ can be moved continuously through infinity without the relative interior of $h_1\delta F$ transversally intersecting a line of $\{L_1, L_2, L_3\}$. During this motion, before these copies of $h_1\delta F$ touch L_4 again, they cannot avoid touching one of the lines of $\{L_1, L_2, L_3\}$, otherwise it would contradict the maximally of h_1. This implies that the disk $h_1\delta F$ is imprisoned by $\{L_1, L_2, L_3\}$; but it is easy to see, using only simple geometric arguments, that this is impossible. Consequently, the lines of Ω intersect $h_1\delta F$. Therefore, any small rigid movement of $h_1\delta F$ maintains $h_1\delta F$ intersecting all lines of Ω. □

Corollary 1 *There is a copy of a convex disk F that is trapped by the lines $\Omega = \{L_1, L_2, L_3, L_4\}$, provided there is a homothetic copy to F imprisoned by Ω.*

Theorem 3 *There is a strictly convex disk trapped by the lines of $\Omega = \{L_1, L_2, L_3, L_4\}$, provided they immobilize a quadrangle or a triangle.*

Proof Let $F \subset \mathbb{R}^2$ be a strictly convex disk. If there is a copy of F trapped by the lines of Ω, by Corollary 1 we may assume without loss of generality that there is $\gamma \in SE(3)$ such that γF is imprisoned by $\Omega = \{L_1, L_2, L_3, L_4\}$. We have two cases to analyze:

(a) the line L_i transversally intersects $\gamma\mathbb{R}^2$ at $x_i \in \text{bd}\gamma F$, $j = 1,2,3,4$, and $\{x_1, x_2, x_3, x_4\}$ immobilize γF in the plane in $\gamma\mathbb{R}^2$, or
(b) the line L_i transversally intersects $\gamma\mathbb{R}^2$ at $x_i \in \text{bd}\gamma F$, $j = 1,2,3$, $L_4 \subset \gamma\mathbb{R}^2$ intersects the relative interior of γF, and $\{x_1, x_2, x_3\}$ immobilize γF in the plane in $\gamma\mathbb{R}^2$.

In the first case let C be the quadrangle whose sides ℓ_i are contained in the support lines of γF at x_i, $i = 1,2,3,4$. In the second case C is a triangle whose sides ℓ_i are contained in the support lines of γF at x_i, $i = 1,2,3$. Note that in both cases C is imprisoned by Ω. We shall now prove that C is immobilized by Ω.

Let Δ be a small compact neighborhood of the identity in $SE(3)$ with the property that for every $\zeta \in \Delta$, ζC intersects the lines of Ω. Let $P = \{\zeta \in \Delta \mid$ the lines of

Ω intersect the boundary of $C\}$. We shall prove that $P = \{\mathrm{Id}\}$, thus proving that C is immobilized.

In the first case, for every $\zeta \in \Delta$, and $i = 1, 2, 3, 4$, let $x_i^\zeta \in \mathrm{bd}C$ be the point with the property that $\zeta x_i^\zeta \in \zeta C$ belongs to L_i. We may assume without loss of generality that $x_i^\zeta \in \ell_i$, $i = 1, 2, 3, 4$. Since $\gamma_0 F$ is imprisoned by Ω, the disk $\zeta\gamma_0 F$ intersects the lines of Ω at the boundary and hence, since this disk is strictly convex, we have that $x_i^\zeta = x_i$ for every $\zeta \in \Lambda$, $i = 1, 2, 3$.

The quadrangle K with vertices $\{x_1, x_2, x_3, x_4\}$ has the property that it can be moved through small rigid movements in such a way that the vertices $\{x_1, x_2, x_3, x_4\}$ are moved along the lines L_1, L_2, L_3 and L_4 respectively. By [5, Theorem 2], either the four lines are parallel or there is a line ℓ that these four lines intersect perpendicularly. Under these circumstances, it is not difficult to check directly that $\gamma_0 F$ is not imprisoned by Ω; in the first case by translating the disk in the direction of the four lines and in the second case by translating the disk away from the line ℓ in the opposite direction. In either case, this is a contradiction. Therefore, we have that $\Lambda = \{\mathrm{Id}\}$, and hence that C is immobilized by Ω.

Similarly, in the second case, the triangle K with vertices $\{x_1, x_2, x_3\}$ can be moved through small rigid movements along the respective lines, while the relative interior of K slides along L_4. Again, this is impossible unless $\Lambda = \{\mathrm{Id}\}$. This proves that C is immobilized by Ω. □

5 Trapping a Convex Disk with 4 Lines

Let D be the circular disk $\{(x, y, z) : x^2 + y^2 \leq 1, z = 0\}$. Take the points $a = (a_x, b_x, 0)$, $b = (b_x, b_y, 0)$, $c = (c_x, c_y, 0)$ on $\mathrm{bd}D$, at mutual distances $2\pi/3$ (measured on $\mathrm{bd}D$).

Notice that, in the plane $z = 0$, D cannot move far from its position without meeting the set $\{(101/100)a, (101/100)b, (101/100)c\}$.

Consider the points

$$a' = ((101/100)a_x, (101/100)a_y, -\varepsilon), b' = ((101/100)b_x, (101/100)b_y, -\varepsilon),$$
$$c' = ((101/100)c_x, (101/100)c_y, 200\varepsilon).$$

For any $\varepsilon > 0$, the line $a'b'$ lies below D and the lines $a'c'$, $b'c'$ above D. For ε very small, the line cc' goes below D and very close to it. So, the lines $a'b'$, $b'c'$, $a'c'$ and cc' trap D.

Theorem 4 *Every convex disk with a C^2 boundary is trapped by 4 lines.*

Proof By Theorem 3 in [4], every convex disk $D \subset \mathbb{R}^2$ with a C^2 boundary can be immobilized in $\mathbb{R}^2$ by 3 suitably chosen points.

By using the same construction as in the example above, we find 4 lines trapping D in $\mathbb{R}^3$. □

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