Hamiltonicity in k-tree-Halin Graphs

Ayesha Shabbir and Tudor Zamfirescu

1 Introduction

All graphs here are undirected, connected and without loops or multiple edges. A *Halin graph* is a graph $H = T \cup C$, where T is a tree with no vertex of degree two, and C is a cycle connecting the leaves of T in the cyclic order determined by a planar embedding of T. (As H has no multiple edges, $T \neq K_2$.) Halin graphs belong to the family of all planar, 3-connected graphs and possess strong hamiltonian properties. One of these is the property of uniform hamiltonicity, which was proven in [7]. A graph is called *uniformly hamiltonian* if each of its edges lies in some hamiltonian cycle and is missed by another one [2] (see also [7]).

For various generalizations of Halin graphs and investigation of their hamiltonian properties, see [3, 5–8]. We present here the following new generalization of Halin graphs.

A *k-tree-Halin graph* is a planar graph $F \cup C$, where F is a forest without vertices of degree 2 and with at most k components, and C is a cycle such that $V(C)$ is the set of all leaves of F and C bounds a face.

Let $\mathcal{G}_k$ be the family of all k-tree-Halin graphs. Then $\mathcal{G}_1$ consists of Halin graphs, which are 3-connected and hamiltonian, while the graphs in $\mathcal{G}_k \backslash \mathcal{G}_1$ have connectivity number 2 ($k \geq 2$). We shall see that the hamiltonicity of k-tree-Halin graphs steadily decreases as k increases. Indeed, a 2-tree-Halin graph is still hamiltonian, a 3-tree-Halin graph is not always hamiltonian but still traceable, while a 4-tree-Halin graph is not even necessarily traceable.

A. Shabbir
Abdus Salam School of Mathematical Sciences, GC University, 68-B, New Muslim Town, Lahore, Pakistan
e-mail: ashinori@hotmail.com

T. Zamfirescu (✉)
Faculty of Mathematics, University of Dortmund, 44221 Dortmund, Germany
e-mail: tuzamfirescu@googlemail.com

T. Zamfirescu
"Simion Stoilow" Institute of Mathematics, Roumanian Academy, Bucharest, Roumanian

K. Adiprasito et al. (eds.), *Convexity and Discrete Geometry Including Graph Theory*, Springer Proceedings in Mathematics & Statistics 148,
DOI 10.1007/978-3-319-28186-5_5

We shall pay special attention to the cubic case. We prove that every cubic 3-tree-Halin graph is hamiltonian and every cubic 5-tree-Halin graph is traceable.

2 Preliminaries

In this section we introduce some notation and several notions. We also present several lemmas which are going to be frequently used to prove the main results of the article.

If a k-tree-Halin graph H is written as $H = F \cup C$, then F and C will always mean the forest and the cycle from its definition.

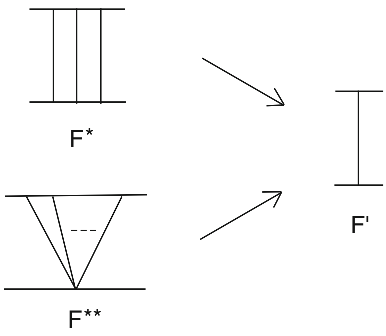
A graph H_x is called a *reduced Halin graph* if it is obtained from a Halin graph $H = F \cup C$ (in this case, F is a tree) by deleting a vertex $x \in C$. The three neighbours of x, whose degrees reduce by one, are called the *end-vertices* of H_x. By deleting from H an edge (x, y) of C we obtain an *edge-reduced Halin graph*, with *end-vertices* x, y.

The following lemma is due to Bondy and Lovasz [1]. It also follows from the uniform hamiltonicity of Halin graphs [7].

Lemma 1 *In any reduced Halin graph each pair of end-vertices is joined by some Hamiltonian path (Fig. 1). Consequently, in any edge-reduced Halin graph, the end-vertices are joined by some hamiltonian path.*

Remark 1 This lemma is a key ingredient of our proving technique, as it implies the following. One can contract any reduced Halin subgraph of a graph G to a single vertex without altering the hamiltonicity or traceability of G.

Remark 2 Let G contain the fragment F^* or F^{**} shown below. Replace this fragment with the fragment F' and obtain a graph G'. If G' is hamiltonian (or traceable) then G is also hamiltonian (or traceable).



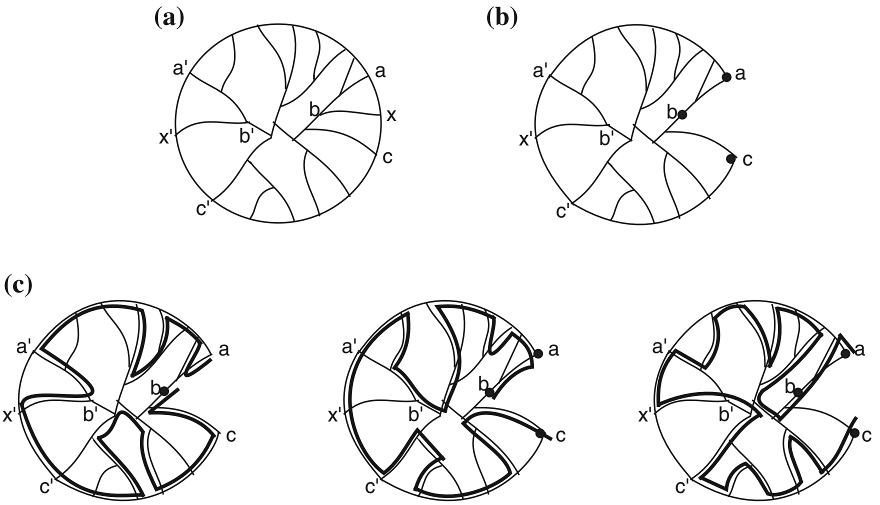


Fig. 1 **a** Cubic Halin graph; **b** Cubic reduced Halin graph; **c** Hamiltonian paths between end-vertices of H_x

A graph is said to be 1-*edge hamiltonian* if for any of its edges, the graph has a hamiltonian cycle using it.

The following lemma is a corollary of Theorem 1 in [4].

Lemma 2 *In any edge-reduced Halin graph H obtained from a cubic Halin graph, for each edge e there exists a hamiltonian path joining the end-vertices of H, which uses the edge e.*

Lemma 3 *Consider the graph K of Fig. 2 consisting of three horizontal pairwise disjoint paths, two pairs of which are joined by vertical edges, as illustrated in the figure. There are 3 left and 3 right end-vertices of the paths. For any pair of left end-vertices there is a unique pair of paths starting there, ending at a pair of right end-vertices, and together spanning K minus the two unused end-vertices. This uniqueness yields a bijection between the triple of pairs of left end-vertices and the triple of pairs of right end-vertices.*

Proof See [4]. □

3 Main Results

A k-tree-Halin graph $F \cup C \notin \mathcal{G}_{k-1}$ with $k \geq 2$ will be called *cyclic* if it cannot be disconnected by deleting any component T_i of F $(i = 1, 2, \ldots, k)$, see Fig. 3. It will be called *strictly linear* if it cannot be disconnected by deleting T_1 or T_k, but

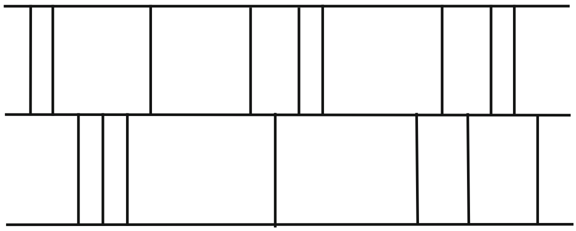


Fig. 2 The graph K

it decomposes in exactly two components by deleting any other component T_i of F $(i = 2, \dots, k-1)$, see Fig. 4; and we will call it *linear* if for any tree $T_i \subset F$, $G - T_i$ has at most two connected components.

Theorem 1 *Every cyclic k-tree-Halin graph is hamiltonian.*

Proof Let $G = F \cup C$ be a cyclic k-tree-Halin graph. By Lemma 1, we can replace every component of F by a single vertex and G becomes a cycle. □

The next result is a consequence of Lemma 2.

Theorem 2 *Every cubic cyclic k-tree-Halin graph is* 1*-edge-hamiltonian.*

Theorem 3 *Every strictly linear k-tree-Halin graph is traceable.*

Proof Let $G = F \cup C$ be a linear k-tree-Halin graph. The statement is true for $G \in \mathcal{G}_2$, by Theorem 1.

Let us prove the result for any $G \in \mathcal{G}_k \backslash \mathcal{G}_{k-1}$ $(k \geq 3)$. In any such graph G, first, by using Lemma 1, we replace the components T_1 and T_k by single vertices on C. Next, we consider the subgraph G_i of G, which is spanned by some T_i $(i = 2, \dots, k-1)$, see Fig. 5a.

By replacing reduced Halin subgraphs of G_i by single vertices we obtain the graph G_i', see Fig. 5b. By Remark 1 the traceability of G_i and that of G_i' are equivalent. By applying the transformations of Remark 2 on G_i' we obtain the graph G_i'', see Fig. 5c. Using the notation of Fig. 5c, G_i'' has the hamiltonian paths (1) $a_1a_2 \dots a_s b_m b_{m-1} \dots b_1 c_1 c_2 \dots c_r$ and (2) $c_1 c_2 \dots c_r b_m b_{m-1} \dots b_1 a_1 a_2 \dots a_s$. For every i, G_i'' has such hamiltonian paths. Using paths analogous to (1) for odd i and analogous to (2) for even i yields a hamiltonian path in G, by Remark 2. □

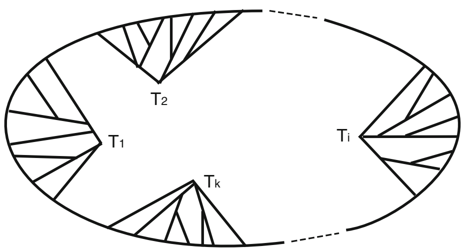


Fig. 3 Cyclic k-tree-Halin graph

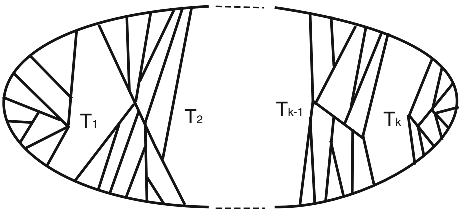


Fig. 4 Strictly linear k-tree-Halin graph

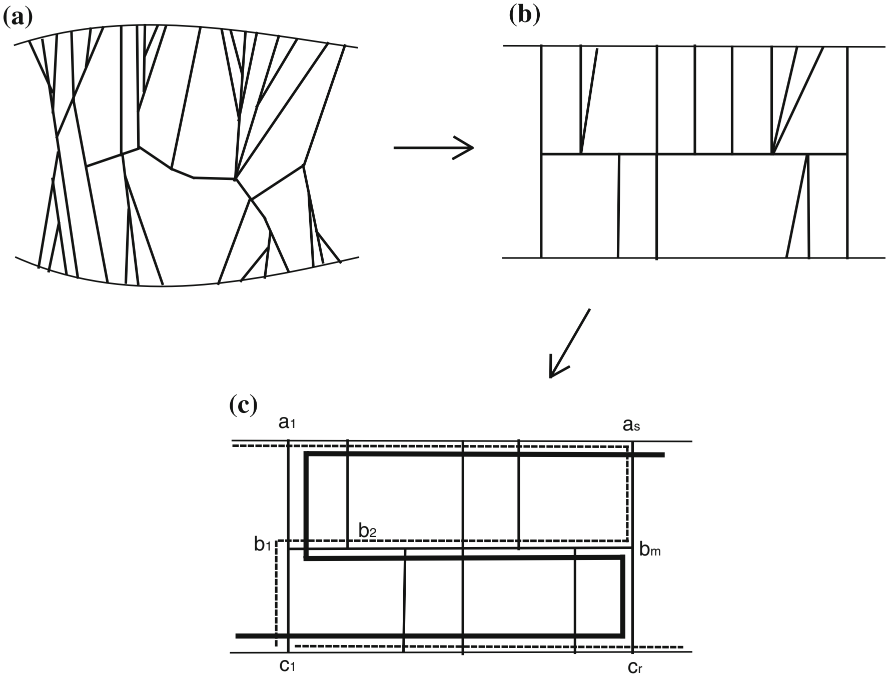


Fig. 5 **a** A subgraph G_i of G; **b** The graph G_i' obtained by replacing reduced Halin subgraphs of G_i by single vertices; **c** Transformation of G_i' into G_i''

Theorem 4 *Every cubic linear k-tree-Halin graph is hamiltonian.*

Proof For $\mathcal{G}_2$, the statement is verified by Theorem 1.

Let $G = F \cup C$ be a cubic linear k-tree-Halin graph such that F has at least three components. Once again by using Lemma 1, we transform each T_i the deletion of which does not disconnect G into single vertices. Next, consider any $T_i \subset F$ whose deletion disconnects G. The subgraph G_i of G, which is spanned by this T_i, after the reductions mentioned in Remark 2 looks like in Fig. 6a. To prove the theorem, it suffices to show that in any such subgraph of G we have a pair of disjoint spanning paths which start from (a_1, c_1) and end in (a_s, c_r). See Fig. 6 and 7.

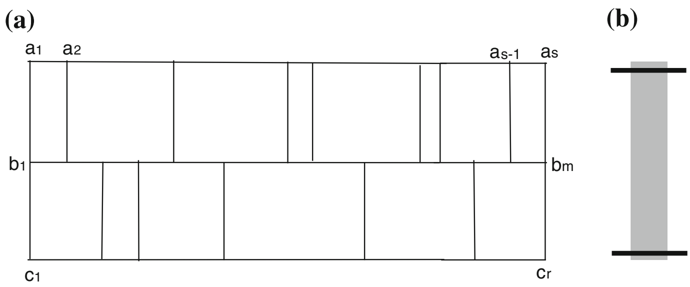
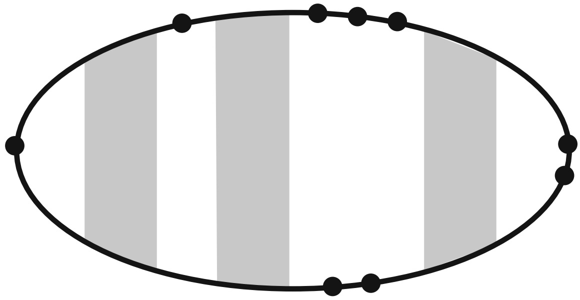


Fig. 6 Subgraph G_i of G, after the reductions mentioned in Remark 2

Fig. 7 Subgraph of G with pair of disjoint spanning paths



The graph G_i minus the edges (a_1, b_1), (b_1, c_1), (a_s, b_m), (b_m, c_r) is a graph K to which we can apply Lemma 3. We find two suitable paths from one of the pairs (a_1, b_1), (b_1, c_1) to one of the pairs (a_s, b_m), (b_m, c_r), and these can be extended to two paths from (a_1, c_1) to (a_s, c_r) which together span G_i. □

The next three theorems are derived from the previous theorems.

Theorem 5 *Every* 2*-tree-Halin graph is hamiltonian. Every cubic* 2*-tree-Halin graph is* 1*-edge hamiltonian.*

Proof Use Theorems 1 and 2. □

Theorem 6 *Every* 3*-tree-Halin graph is traceable but not necessarily hamiltonian. Not all* 4*-tree-Halin graphs are traceable.*

Proof The graph shown in Fig. 8a is a non-hamiltonian 3-tree-Halin graph, while the traceability of any 3-tree-Halin graph follows from Theorem 3.

When passing to $\mathcal{G}_4$ the traceability is lost. In Fig. 8b, we present a non-traceable 4-tree-Halin graph. □

Theorem 7 *Every cubic* 3*-tree-Halin graph is hamiltonian. There exist non-hamiltonian cubic* 4*-tree-Halin graphs.*

Proof This follows from Theorem 4. For a non-hamiltonian cubic 4-tree-Halin graph, see Fig. 8c. □

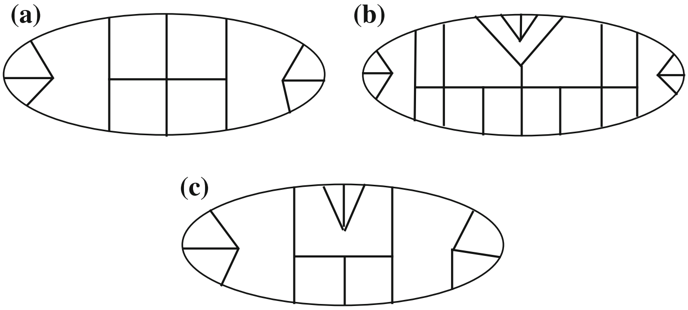


Fig. 8 **a** A non-hamiltonian 3-tree-Halin graph. **b** A non-traceable 4-tree-Halin graph. **c** A cubic non-hamiltonian 4-tree-Halin graph

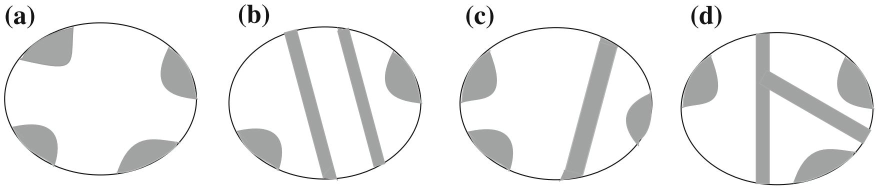


Fig. 9 Four possible types of $G \in G_4 \backslash G_3$

Lemma 4 *Every cubic* 4*-tree-Halin graph is traceable.*

Proof If F has up to 3 components, then we are done, by Theorem 7. So we are left with the case when F has exactly 4 components.

Any $G \in \mathcal{G}_4 \backslash \mathcal{G}_3$ is of one of the four possible types (A), (B), (C), (D) shown in Fig. 9, where each shaded region represents a tree. If G is of type (A), (B) or (C), its traceability follows from Theorem 4.

Let $G \in \mathcal{G}_4 \backslash \mathcal{G}_3$ be cubic, of type (D), and assume that $G - T_4$ has 3 components. By Lemma 1, we replace T_1, T_2 and T_3, respectively, by the vertices v_1, v_2 and v_3 on C. Next, we consider the subgraph spanned by T_4 in G and apply all possible reductions mentioned in previous proofs. The resulting graph G' looks like in Fig. 10a or b.

In the case of Fig. 10a, a hamiltonian path in G' is shown as well.

To prove the traceability in G' when it appears according to Fig. 10b, we proceed as follows.

Let H_1 be the component of $G' - (a_1 \cup b_2 \cup b_3)$ containing v_1. By Lemma 3, a hamiltonian path of G which visits v_1 without having an end-vertex in H_1 can use only two of the three pairs of edges (a_1, b_2), (b_2, b_3), (b_3, a_1). At least one of them contains a_1; w.l.o.g. it is (a_1, b_2). The same argument applied to H_3 yields that some hamiltonian path of H_3 can be used to join b_2 to a vertex of H_3 incident to a_3 or b_1. Also, the argument applied to H_2 exhibits a hamiltonian path of H_2 joining a_2 to a

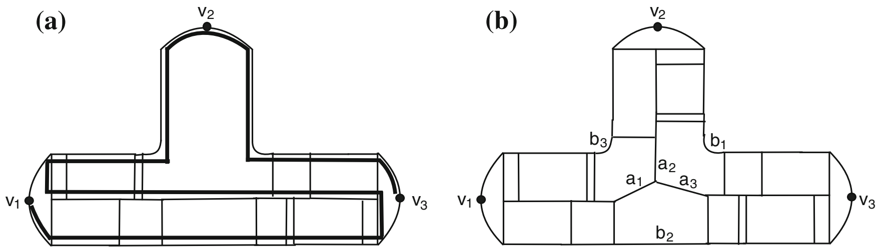


Fig. 10 Graph G' with all possible reductions applied

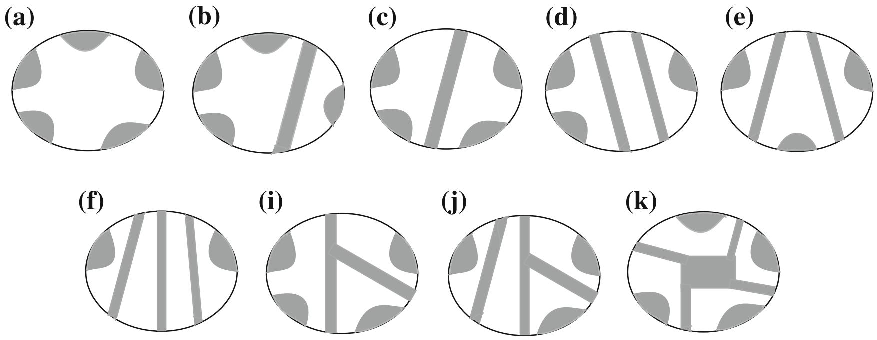


Fig. 11 Traceable cubic 5-tree-Halin graphs: the last case needs special treatment

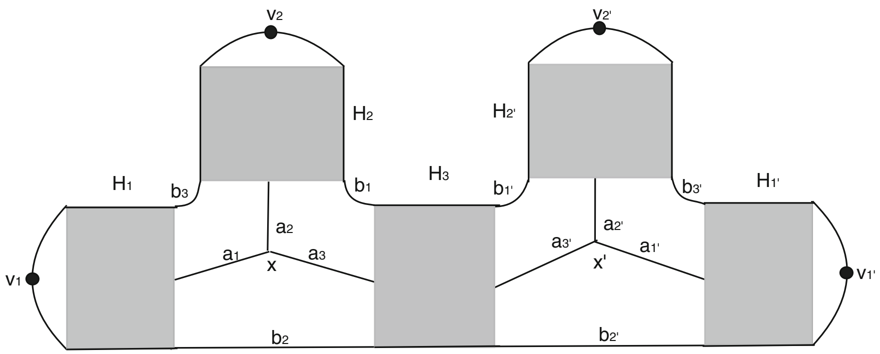


Fig. 12 Graph G with all reductions applied

vertex of H_2 incident to b_3 or b_1. Then these hamiltonian paths in H_2, H_1, H_3 yield a hamiltonian path in G' and consequently in G. □

Theorem 8 *Every cubic* 5*-tree-Halin graph is traceable. There exist non-traceable cubic* 7*-tree-Halin graphs.*

Proof Due to Lemma 4, it suffices to prove the first part of the statement for any cubic $G \in \mathcal{G}_5 \backslash \mathcal{G}_4$. Any such G can be of one of the types (A) to (K) shown in Fig. 11.

The traceability of G when it is of type (A), (B), (C), (D), (E), (F), (I) or (J), can be checked by using previous results. Type (K) needs special treatment.

Let $G \in \mathcal{G}_5 \backslash \mathcal{G}_4$ be of type (K), and T_5 be the tree whose removal makes G disconnected into exactly four components. After applying all previously mentioned reductions, we get a graph, which we also call G, see Fig. 12. This graph minus x, x' and the edges $b_1, b_2, b_3, b_1', b_2', b_3'$ has five components, $H_1, H_2, H_1', H_2', H_3$, see Fig. 12. If there exists a hamiltonian path in H_1 joining two of the edges a_1, b_2, b_3, we say that those edges *serve* H_1 (or simply *serve*), and use an analogous language about the other components H_2, H_1', H_2', H_3, too.

Case 1. (b_1, b_3) *or* (b_2, b_3) *serves, and* (b_1', b_3') *or* (b_2', b_3') *serves.*

W.l.o.g. assume that (b_1, b_3) and (b_2', b_3') serve. Let D_2 be the hamiltonian path of H_2 joining b_1 to b_3, and D_1' the hamiltonian path of H_1' joining b_2' to b_3'. Let yC_iz or $yC_i^{'}z$ be the path in C joining $y, z \in E(C)$ through H_i or H_i'. We consider the cycle

$$\Gamma = b_2C_1b_3D_2b_1C_3b_1^{'}C_2^{'}b_3^{'}D_1^{'}b_2^{'}C_3b_2.$$

Then $G - \Gamma$ is a path P (containing x, x'). To obtain a hamiltonian path in G, we simply open up Γ by deleting an edge of it incident to a neighbour of v_1, but not to v_1, and joining that neighbour to an end-vertex of P.

Case 2. Neither (b_1, b_3), *nor* (b_2, b_3) *serves, and neither* (b_1', b_3'), *nor* (b_2', b_3') *serves.*

In this case (a_2, b_1) serves H_2, (a_1, b_2) serves H_1, (a_2', b_1') serves H_2' and (a_1', b_2') serves H_1'. Let D_2, D_1, D_2', D_1' be the resulting hamiltonian paths. Consider now the cycle

$$\Gamma = b_2D_1a_1xa_2D_2b_1C_3b_1^{'}D_2^{'}a_2^{'}x^{'}a_1^{'}D_1^{'}b_2^{'}C_3^{'}b_2.$$

Then $G - \Gamma$ is a path P joining a vertex adjacent to x with a vertex adjacent to x'. As above, by opening up Γ at a suitable vertex incident to b_2, and joining to P, we obtain a hamiltonian path of G.

Case 3. (b_1, b_3) *or* (b_2, b_3) *serves, but neither* (b_1', b_3'), *nor* (b_2', b_3') *serves.*

Assume again w.l.o.g. that (b_1, b_3) serves, and use the path D_2 from Case 1, and the paths D_2', D_1' from Case 2. Consider the cycle

$$\Gamma = b_2C_1b_3D_2b_1C_3b_1^{'}D_2^{'}a_2^{'}x^{'}a_1^{'}D_1^{'}b_2^{'}C_3b_2.$$

Then $G - \Gamma$ is a path P. We again open Γ up at a suitable neighbour of v_1, and join with P.

Hence, G is traceable.

The graph shown in Fig. 13, is a cubic non-traceable 7-tree-Halin graph. □

Conjecture. Every cubic 6-tree-Halin graph is traceable.

Acknowledgments Thanks are due to the referees of this paper. The second author's work was supported by a grant of the Roumanian National Authority for Scientific Research, CNCS—UEFISCDI, project number PN-II-ID-PCE-2011-3-0533.

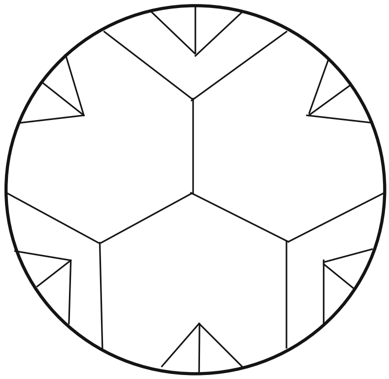


Fig. 13 A non-traceable cubic 7-tree-Halin graph

References

1. J.A. Bondy, L. Lovasz, Length of cycles in Halin graphs. J Graph Theory **8**, 397–410 (1985)
2. C.A. Holzmann, F. Harary, On the tree graph of a matroid. SIAM J Appl Math **22**, 187–193 (1972)
3. S. Malik, A.M. Qureshi, T. Zamfirescu, Hamiltonian properties of generalized Halin graphs. Can Math Bull **52**(3), 416–423 (2009)
4. S. Malik, A.M. Qureshi, T. Zamfirescu, Hamiltonicity of cubic 3-connected k-Halin graphs. Electronic J Combin **20**(1), 66 (2013)
5. M. Skowrońska, Hamiltonian properties of Halin-like graphs. Ars Combinatoria **16-B**, 97–109 (1983)
6. M. Skowrońska, M.M. Sysło, Hamiltonian cycles in skirted trees. Zastosow Mat **19**(3–4), 599–610 (1987)
7. Z. Skupień, *Crowned trees and planar highly hamiltonian graphs* (Wissenschaftsverlag, Mannheim, Contemporary methods in Graph theory, 1990), pp. 537–555
8. C.T. Zamfirescu, T.I. Zamfirescu, Hamiltonian properties of generalized pyramids. Math Nachr **284**, 1739–1747 (2011)