Six Problems on the Length of the Cut Locus

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Introduction Let S be a compact convex surface in $\mathbb{R}^3$, with intrinsic metric ρ and intrinsic diameter 2.

A *segment ab* is a shortest path on S from a to b (of length $\rho(a, b)$).

Let $M \subset S$ be compact. A point $x \in S$, such that some shortest path xy from x to M, called a *segment from x to M*, cannot be extended as a shortest path to M beyond x, is called a *cut point* with respect to M in direction of yx. Moreover, the set $C(M)$ of all cut points with respect to M is called the *cut locus* of M. If M contains a single point x, we write $C(x)$ for $C(M)$. Let λ denote the *length*, i.e. 1-dimensional Hausdorff measure.

It is known that cut loci are local trees [5], even trees if card $M = 1$. We are looking for bounds for the length of the cut locus.

Take $M = \{x\}$. It was shown in [2] and [5] (and it already followed from [7]) that $\lambda C(x)$ may be infinite. $C(x)$ may even fail to have locally finite length: there are convex surfaces S on which, for any point x, every open set in S contains a compact subset of $C(x)$ with infinite length [8]. Even if in the Riemannian case this cannot happen (see [1, 2]), $\lambda C(M)$ still has no upper bound depending only on card M.

The case when the surface S is a sphere shows that the lower bound vanishes.

So, which bounds do we want to discover?

Polyhedral surfaces (Vîlcu). We restrict now the study to the surface S of a convex polytope, still of diameter 2, and to sets M of cardinality 1, when cut loci enjoy very

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nice properties. See e.g. [3] and the references therein. For example, $C(x)$ contains all vertices of S (excepting x, if x is a vertex), and its leaves are vertices of S. The *ramification points* of $C(x)$, which are the points $v \in C(x)$ of degree $d(v) \geq 3$, are joined to x by precisely $d(v)$ segments. The graph edges of $C(x)$ (here regarded as a 1-dimensional complex) are segments on S.

Notice first that the upper bound for $\lambda C(x)$ cannot be achieved at a vertex x of S, because points close enough to x have a longer cut locus.

Consider a tetrahedron $T_\varepsilon = abcx$ with $\lambda ab = \lambda bc = \lambda ca = \varepsilon$. Then $\lambda C(x) = \varepsilon\sqrt{3}$ on T_ε; hence, the lower bound for $\lambda C(x)$ is zero if x is allowed to be a vertex.

We can give now four problems, originating from a procedure of flattening convex polyhedral surfaces, and mainly based on [3].

1. Give lower and upper bounds for $\lambda C(x)$, where $x \in S$ is not a vertex of S.
2. Locate on S a point x for which $C(x)$ has minimal length.
3. Locate on S a point x with minimal number of ramification points for $C(x)$.
4. Characterize S such that, for some $x \in S$, $C(x)$ has precisely one ramification point. How many such points x may exist?

Arbitrary convex surfaces (Zamfirescu). If card $M = 2$, the lower bound for $\lambda C(M)$ is also zero: take S to be an ellipsoid of revolution with two axes of arbitrarily small length, and take M to consist of the two endpoints of the long axis.

At the Mulhouse Conference on Convex and Discrete Geometry (7 – 11 September 2014), the second author recalled the conjecture in [4] from 2005, saying that $\lambda C(M) \geq 1$ whenever $3 \leq \text{card} M < \aleph_0$ and S is smooth, and announced that the case card $M = 3$ was solved. Now he announces that the conjecture is proven, for any compact convex surface S [6].

But, for infinite M, the bound vanishes again! Take M to be a great circle on a sphere.

And now the last two problems:

5. Find a lower bound for the length of the cut locus of a countably infinite compact set.
6. Find a lower bound for the length of the cut locus of a Jordan arc (i.e., of a topological line-segment).

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