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Right Triple Convexity

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Dedicated to the memory of Jean Jacques Moreau.

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A set M in $\mathbb{R}^d$ is *rt*-convex if every pair of its points is included in a 3-point subset $\{x, y, z\}$ of M satisfying $\angle xyz = \pi/2$.

We characterize *rt*-convex sets, and investigate *rt*-convexity for 2-connected polygonally connected sets, for 3-connected sets, for geometric graphs, and for finite sets.

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Introduction

At the meeting on Convexity in Oberwolfach from 1974, the second author proposed the investigation of the following convexity concept: Given a family $\mathcal{F}$ of sets in a certain space X, a set $M \subset X$ is called $\mathcal{F}$*-convex* if for any pair of distinct points $x, y \in M$ there is a set $F \in \mathcal{F}$ such that $x, y \in F$ and $F \subset M$. This appeared explicitly as a problem, on that occasion.

It is easily seen that usual convexity, affine linearity, arc-wise connectedness, polygonal connectedness, are all examples of $\mathcal{F}$-convexity (for suitably chosen families $\mathcal{F}$).

Blind, Valette and the second author [1], and later Böröczky Jr. [2], investigated the rectangular convexity, the case when $\mathcal{F}$ contains all non-degenerate rectangles, but a conjectured characterization remained unproved.

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Among the more recent investigations, let us mention Magazanik and Perles' staircase connectedness, a special kind of polygonal connectedness [3].

In [8] the second author studies the $\mathcal{F}$-convex sets when $\mathcal{F}$ is the family of all right triangles in a Hilbert space of dimension at least 2. A *right triangle* is the convex hull of 3 distinct points x, y, z with $\angle xyz = \pi/2$. Those $\mathcal{F}$-convex sets are all convex, because so are the elements of $\mathcal{F}$.

In [8] it is also mentioned that "this $\mathcal{F}$-convexity is a special case of $\mathcal{F}'$-convexity, where $\mathcal{F}'$ is the family of all triples $\{x, y, z\}$ such that $\angle xyz = \pi/2$. The (interesting) study of $\mathcal{F}'$-convexity includes a more discrete geometric research, while the $\mathcal{F}$-convexity, called *right convexity*, is fully embedded in convex geometry."

As usual, for $M \subset \mathbb{R}^d$ with $d \geq 2$, $\overline{M}$ denotes its topological closure, and $\operatorname{diam} M = \sup_{x,y \in M} \|x - y\|$. A 2-point set $\{x, y\} \subset M$ with $\|x - y\| = \operatorname{diam} M$ is called a *diametral pair* of M, while the line-segment xy is a *diameter* of M.

Let $M \subset \mathbb{R}^d$. A pair of points in M is said to enjoy *the rt-property in M* if it is included in a set $\{x, y, z\} \subset M$ such that $\angle xyz = \pi/2$

A set $M \subset \mathbb{R}^d$ is called *right-triple-convex*, for short *rt-convex*, if any pair of its points enjoys the *rt*-property. Clearly, this *rt*-convexity generalizes the concept of right convexity introduced in [8].

A set in $\mathbb{R}^d$ is called *polygonally connected* if any pair of its points can be joined by a polygonal line included in the set.

A continuum, i.e. a compact connected set, C is said to be *n-connected* if for any subset $F \subset C$ with $\operatorname{card} F \leq n - 1$, the set $C \setminus F$ is connected.

A set $M \subset \mathbb{R}^d$ is called *almost rt-convex* if each pair of points of M, with at most one exception, enjoys the *rt*-property.

For any compact set $C \subset \mathbb{R}^d$, let S_C be the smallest hypersphere containing C in its convex hull.

For distinct $x, y \in \mathbb{R}^d$, let l_{xy} be the line through x, y and H_{xy} be the hyperplane through x orthogonal to l_{xy}. Also, put $W_{xy} = (S_{xy} \cup H_{xy} \cup H_{yx}) \setminus \{x, y\}$.

We shall denote by μ_n the n-dimensional Hausdorff measure.

rt-convexity of 2-connected polygonally connected continua.

Let $\mathcal{A}$ be the space of all compact 2-connected polygonally connected sets in $\mathbb{R}^d$.

Proposition. *Let σ be a diametral pair in $K \in \mathcal{A}$. Then every pair of points in K different from σ has the rt-property.*

Proof. Let $\sigma = \{u, v\}$, and let $x, y \in K$ verify $\{x, y\} \neq \sigma$.

Either uv is another diameter of the hypersphere S_{xy}, or at least one of the points u and v, say v, lies outside S_{xy}.

If uv is another diameter of S_{xy}, then $\angle xvy = \pi/2$.

If v lies outside S_{xy}, there are four possibilities:

1) $v \in H_{xy} \cup H_{yx}$.
2) v lies between H_{xy} and H_{yx}.
3) v is separated by H_{xy} from y.
4) v is separated by H_{yx} from x.

In case 1) we have $\angle vxy = \pi/2$ or $\angle vyx = \pi/2$.

In case 2), let $P \subset K$ be a polygonal line joining v to x. Obviously P meets $S_{xy} \cup H_{xy}$ not only in x, or only in y and x. So, there is a point $s \in P \cap S_{xy} \setminus \{x, y\}$ and $\angle xsy = \pi/2$, or there is a point $h \in P \cap H_{xy} \setminus \{x\}$ and $\angle hxy = \pi/2$, or else there is a point $h' \in P \cap H_{yx} \setminus \{y\}$ and $\angle h'yx = \pi/2$.

In case 3), $K \setminus \{x\}$ being connected and containing both v and y, it must meet H_{xy} at some point h, whence $\angle hxy = \pi/2$.

Case 4) is analogous to 3).

Corollary 1. *A set $K \in \mathcal{A}$ is rt-convex if and only if some diametral pair σ of K is seen from some point in $K \setminus \sigma$ under a right angle.*

The non-trivial implication follows by choosing the diametral pair σ from the hypothesis and using the Proposition with this σ.

Theorem 1. *Each set $K \in \mathcal{A}$ is almost rt-convex. If it is not rt-convex, then the exceptional pair without the rt-property is diametral.*

Theorem 1 follows directly from the Proposition, while Theorem 2 is obtained by taking successively two different diametral pairs to play the role of σ.

Theorem 2. *If a set $K \in \mathcal{A}$ has more than one diametral pair, then it is rt-convex.*

Convex bodies, i.e. compact convex sets with non-empty interior, also belong to $\mathcal{A}$.

Corollary 2. *Every convex body of constant width is rt-convex*[8].

Corollary 1 and Theorem 2 generalize Theorems 1 and 4 from [8], while our next result generalizes Theorem 2 from [8].

Theorem 3. *A set $K \in \mathcal{A}$ is rt-convex if and only if* $\mathrm{card}(K \cap S_K) \geq 3$.

Proof. It is well-known that $\mathrm{card}(K \cap S_K) \geq 2$. If $\mathrm{card}(K \cap S_K) = 2$, then $K \cap S_K$ is the only diametral pair of K and no other point of K sees it under a right angle. So, by Corollary 1, K is not *rt*-convex.

Conversely, suppose K is not *rt*-convex. By Theorem 1, K has a single diametral pair $\{x, y\}$ which is exceptional, i.e. $K \cap W_{xy} = \emptyset$.

We claim that $S_{xy} = S_K$. Indeed, assume $u \in K$ lies outside S_{xy}. There exists a polygonal line $P \subset K$ joining u to x. Notice that u lies strictly between the hyperplanes through x and y orthogonal to xy, because $\|x - u\| \leq \|x - y\|$ and $\|y - u\| \leq \|x - y\|$. But the existence of P, disjoint from W_{xy}, is impossible.

Thus, $S_{xy} = S_K$ and $\mathrm{card}(K \cap S_K) = 2$. $\square$

Lemma 1 ([4]). *For most convex bodies K (in the sense of Baire categories),* $\mathrm{card}(K \cap S_K) = d + 1$.

Since $d \geq 2$, from Theorem 3 and Lemma 1 it follows that most convex bodies are *rt*-convex. But more than this is true.

Lemma 2. *For all convex bodies $K \subset \mathbb{R}^d$, except those in a nowhere dense subset,* $\mathrm{card}(K \cap S_K) \geq d + 1$.

The assertion of Lemma 2 was not explicitly stated, but proved in [4] for $d = 2$; the extension to higher dimensions presents no special difficulty.

Theorem 4. *All convex bodies except those of a nowhere dense family are rt-convex.*

Proof. By Lemma 2, those convex bodies $K \subset \mathbb{R}^d$ for which $\mathrm{card}(K \cap S_K) \leq d$ form a nowhere dense family. As $d \geq 2$, by Theorem 3, the assertion is true. $\square$

For an alternative proof of Theorem 4, see Theorem 12 in [8].

rt-convexity of 3-connected continua.

We can renounce the condition of polygonal connectedness, but have to pay for this. First, the set should be 3-connected, not just 2-connected. And second, we can only work in the Euclidean plane.

Theorem 5. *Every 3-connected continuum in the plane is almost rt-convex.*

Proof. Let $\{u, v\}$ be a diametral pair of the 3-connected continuum C. If every other pair enjoys the *rt*-property, we are done. If not, let x, y be a pair not enjoying the *rt*-property. At least one of the points u, v, say v, lies outside the circle S_{xy} of diameter xy, otherwise uv must be another diameter of S_{xy} because $\|u - v\| \geq \|x - y\|$, and this yields $\angle xvy = \pi/2$.

If some point of C is separated from y by the line H_{xy}, then $C \setminus \{x\}$ meets H_{xy}, and if z lies in their intersection, $\angle zxy = \pi/2$, and we obtain a contradiction. Since C is 3-connected, it is included in one of those two unbounded components of the complement of $\overline{W_{xy}}$, which are not half-planes. Call this component E.

Now, let $x', y' \in C$ be both distinct from x, y. Then either $H_{x'y'}$ separates one of the points x, y from y', or $H_{y'x'}$ separates one of the points x, y from x'. Thus, there exists a point $z' \in C$ yielding a right triangle $\mathrm{conv}\{x', y', z'\}$.

Consider now a pair $\{x, y'\} \subset C$. Since $S_{xy'} \setminus \{x\}$ separates x from y in E and $C \setminus \{y'\}$ is connected, there must be some point $z \in C \cap S_{xy'}$ different from y', whence $\angle xzy' = \pi/2$. □

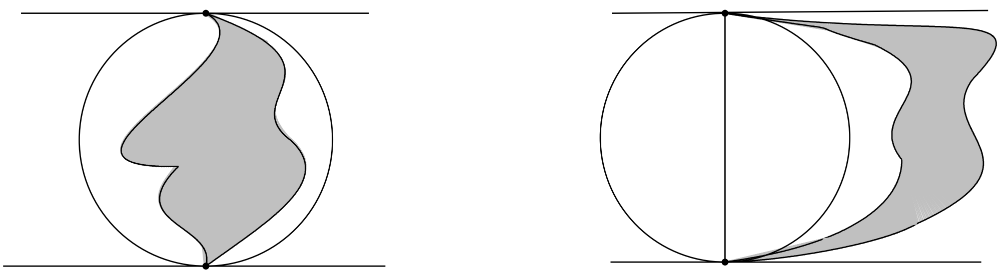


Figure 1:

Example 1. Contrary to the situation in Theorem 1, in Theorem 5 two kinds of exceptional pairs can occur, see Fig. 1.

Theorem 5 cannot be extended to higher dimensions. A counterexample is an ellipsoid with pairwise distinct axis lengths.

rt-convexity of geometric graphs.

We consider in this section *geometric graphs*, which are finite unions of line-segments in $\mathbb{R}^d$. Starting with an abstract finite graph G, with $V(G)$ and $E(G)$ as vertex- and edge-set, respectively, we take $V(G)$ to be a set in $\mathbb{R}^d$, and each edge a line-segment joining its incident vertices, such that any two such line-segments meet in at most one point which is a vertex for both. So we obtain the geometric graph $G^* = \cup\{e : e \in E(G)\} \subset \mathbb{R}^d$. Edges do not cross. We identify G with G^*. In this section, all graphs are geometric graphs.

Let G be such a graph. Whenever it is connected, G is a polygonally connected continuum. As a continuum, G cannot be 3-connected. But 2-connectedness as a graph and as a continuum are equivalent notions.

Let $\mathcal{G}$ be the space of all 2-connected geometric graphs. So, by Theorem 1, each graph of $\mathcal{G}$ is almost *rt*-convex. How scarce are *rt*-convex graphs among all graphs in $\mathcal{G}$?

Endow $\mathcal{G}$ with the Pompeiu-Hausdorff metric h.

Theorem 6. *The set of all rt-convex graphs is dense in* $\mathcal{G}$.

Proof. Let $G \in \mathcal{G}$ and $\varepsilon > 0$. We look for an *rt*-convex graph G' with $h(G, G') < \varepsilon$.

Choose the diametral pair $\{u, v\} \subset G$. If the hypersphere S_{uv} meets G in any third point, then take $G' = G$.

If not, then choose the edge $wv \in E(G)$ such that $\angle uvw$ be maximal. Take a new vertex v^* in the half-circle of S_{uv} containing u, w, v, at distance less than ε from v, and add the new edges v^*v and v^*w, which is allowed, as $v^*v \cap G = \{v\}$ and $v^*w \cap G = \{w\}$ Obviously, the new graph $G' \in \mathcal{G}$ obtained this way satisfies $h(G, G') < \varepsilon$. Moreover, $\angle uv^*v = \pi/2$. By Corollary 1, G' is *rt*-convex. □

The infinite square lattice graph $\mathcal{L}$ in $\mathbb{R}^2$ admits a natural kind of 2-connected geometric subgraphs called grid graphs, defined as follows.

Take in $\mathcal{L}$ some (finite) cycle C, considered as a geometric graph, and consider the geometric graph, called *grid graph*, the vertices and edges of which are all vertices and edges lying on C or inside the bounded plane region of boundary C.

Being 2-connected, each grid graph is, by Theorem 1, almost *rt*-convex. Can we again find (rather) close to any grid graph an *rt*-convex grid graph? The answer is no, not necessarily very close. More precisely, we have the following.

Theorem 7. *If G is a grid graph of order n in $\mathcal{L}$, then there exists an rt-convex grid graph $G' \supset G$ of order at most $\frac{3n}{2} + 2$.*

Proof. The only grid graph on $n < 12$ vertices which is not *rt*-convex is easily found, see Fig. 2, and addition of one vertex and two edges suffices to render it *rt*-convex. So, assume $n \geq 12$.

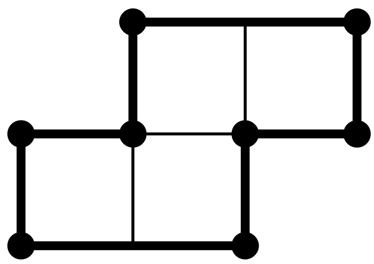


Figure 2:

If G itself is *rt*-convex, we take $G' = G$. If not, consider a diametral pair of G, and assume w.l.o.g. that $\{(0,0), (v,u)\}$ is this pair, where $v \geq u > 0$. (It is easily seen that $u = 0$ and $u = 1$ are impossible.) By Theorem 1, it will be enough to arrange that a diametral pair have the *rt*-property.

Let Γ be the boundary cycle of G. From $(0,0)$ to (v,u) there are two paths $P_1, P_2 \subset \Gamma$ which only meet at their endpoints. The length of $\Gamma = P_1 \cup P_2$ does not exceed n, of course, but $\lambda P_i \geq u + v$ $(i = 1, 2)$. Hence $n \geq 2(u + v)$.

Clearly, the path $(v-1, u)(v, u)(v, u-1)$ is included in Γ.

Consider the path

$$Q = (v,u)(v+1,u)(v+2,u)...(v+u,u)(v+u+1,u)(v+u+1,u-1)$$
$$...(v+u+1,0)(v+u,0)(v+u,1)$$
$$...(v+u,u-1)(v+u-1,u-1)...(v,u-1).$$

The path Q meets G in $\{(v,u),(v,u-1)\}$ and has length $4u+1$.

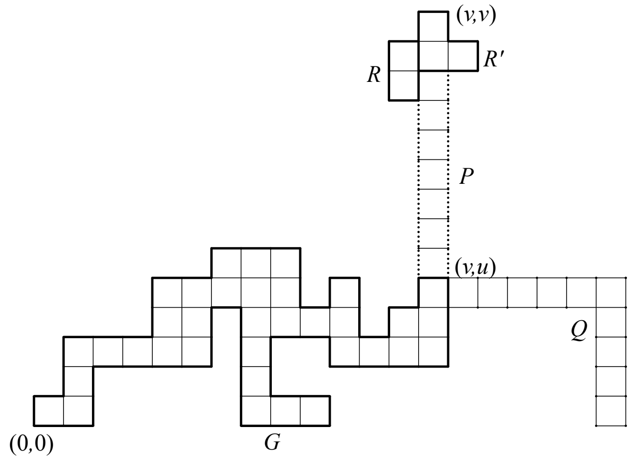


Figure 3:

Let us define the paths

$$R = (v-1,v-3)(v-2,v-3)(v-2,v-2)(v-2,v-1)$$
$$(v-1,v-1)(v-1,v-2)(v,v-2)$$

and

$$R' = (v-1,v-3)(v-1,v-2)(v-1,v-1)(v-1,v)$$
$$(v,v)(v,v-1)(v+1,v-1)(v+1,v-2)(v,v-2).$$

of length $\lambda R = 6$, $\lambda R' = 8$.

Now, let $w = \max\{x : (v-1,x) \in V(G)\}$, and $w' = \max\{x : (v-2,x) \in V(G)\}$.

We assume $w' < v-3$.

We define a path P as follows.

If w equals u or $u+1$, set

$$P = (v-1,w)(v-1,w+1)...(v-1,v-4)R(v,v-3)...(v,u).$$

The length of P is $2(v-u)+1$, respectively $2(v-u)$.

If $u+2 \le w \le v-4$, set

$$P = (v-1,w)(v-1,w+1)...(v-1,v-4)R(v,v-3)$$
$$...(v,w)(v,w-1)(v-1,w-1).$$

Now the length of P is less than $2(v-u)$.

In case $w' \geq v-3$, we replace R by R' in the above definitions of P, thus increasing its length by 2.

This works as long as $u \leq v-2$, $w \leq v-3$. Even for $u = v-2$, and $w = v-1$ or $w = v-2$, we only take off R' the vertex $(v-1, v-3)$ or both $(v-1, v-3)$, $(v-1, v-2)$.

We are left with the cases $u = v-1$ and $u = v$. If $u = v-1$, then $w = v-1$, otherwise G would be *rt*-convex, and take

$$P = (v-1, v-1)(v-1, v)(v-1, v+1)$$
$$(v, v+1)(v, v)(v+1, v)(v+1, v-1)(v, v-1),$$

which adds only 6 extra vertices to G.

If $u = v$, either $(v-2, v)(v-1, v) \in E(G)$, or $(v, v-1)(v, v-2) \in E(G)$, or both hold. Assume w.l.o.g. that $(v, v-1)(v, v-2) \in E(G)$, Then we take

$$P = (v, v-1)(v+1, v-1)(v+1, v-2)(v, v-2).$$

In all cases, $G \cup P$ determines a grid graph with $\{(0,0), (v, v-2)\}$ or $\{(0,0), (v, v-1)\}$ or $\{(0,0), (v+1, v-1)\}$ as a diametral pair, with the *rt*-property.

If $w = v-2$, then $u = v-2$ or $u = v-1$.

Indeed, $u \leq v-4$ implies, for $v \geq 6$,

$$\|(v, u)\| \leq \|(v, v-4)\| < \|(v-1, v-2)\|,$$

which is not true. If $v \leq 5$, then $u \leq v-4 \leq 1$, which is also impossible.

Similarly, $w \geq v-1$ implies $u = w$ or $u = w-1$. The cases $u \geq v-3$ have already been treated.

If $u < v/3$, we use Q, and $G \cup Q$ determines a grid graph with at most $n + 4u$ vertices, so

$$\frac{\operatorname{card} V(G \cup Q)}{\operatorname{card} V(G)} \leq 1 + \frac{4u}{n} < \frac{3}{2},$$

since $n \geq 2(v+u) > 8u$. This graph is *rt*-convex because $(v+u, 0) \in V(G \cup Q)$ sees the diametral pair $\{(0,0), (v+u, u)\}$ under a right angle.

If $u \geq v/3$, we use the path P, and the *rt*-convex grid graph determined by $G \cup P$ has at most $\max\{n+8, n+2(v-u)+2\}$ vertices. If this maximum is $n+8$, then

$$\frac{\operatorname{card} V(G \cup P) - 2}{\operatorname{card} V(G)} \leq 1 + \frac{6}{n} \leq \frac{3}{2},$$

because we assumed $n \geq 12$. Otherwise,

$$\frac{\operatorname{card} V(G \cup P) - 2}{\operatorname{card} V(G)} \leq 1 + \frac{2(v-u)}{n} \leq \frac{3}{2},$$

because $v - u \leq 2v/3$ and $n \geq 2(v+u) \geq 8v/3$. $\square$

rt-convexity of starshaped sets

We now investigate the rt-convexity of starshaped sets.

Let $\mathcal{S}$ be the space of all compact starshaped sets in $\mathbb{R}^d$, where $d \geq 2$. Let $\ker K$ be the kernel of K. For $K \in \mathcal{S}$, let $\operatorname{ex} K$ be the set of all points $x \in K$ such that $kx \subset ky$ and $k \in \ker K$ imply $y = x$.

Recall

$$W_{xy} = (S_{xy} \cup H_{xy} \cup H_{yx}) \setminus \{x, y\},$$

for $x \neq y$. Also, for $x \in \mathbb{R}^d$ and $M \subset \mathbb{R}^d$, and for an affine subspace L of $\mathbb{R}^d$, denote by $p_L(x)$ and $p_L(M)$ the orthogonal projections of $a \in \mathbb{R}^d$ and $M \subset \mathbb{R}^d$ onto L, and put

$$D_x(M) = \left\{ \frac{y - x}{\|y - x\|} : y \in M \setminus \{x\} \right\}.$$

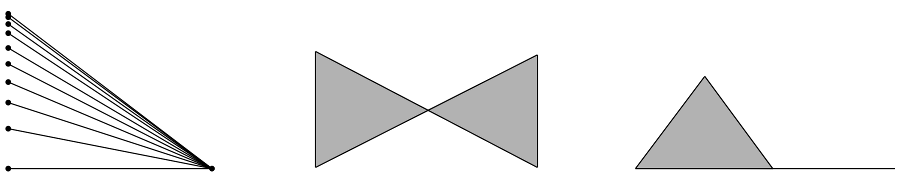
Let us say that a compact set K *looks at least half-dense from* $x \in K$ if there is a closed half-space H^+ with the origin $\mathbf{0}$ on its boundary, such that for any neighbourhood N of x, the set $D_x(N \subset K)$ is dense in $S^{d-1} \cap H^+$, where S^{d-1} is the unit sphere in $\mathbb{R}^d$.

Theorem 8. *If $K \in \mathcal{S}$ is rt-convex, then $\mu_1(K) = \infty$.*

Proof. Let $k \in \ker K$ and $x \in K$ be farthest from k. Choose $y_1 \in kx$. Since x, y_1 have the rt-property, there exists a point $z_1 \in K \cap W_{xy_1}$. As $\|k - z_1\| \leq \|k - x\|$, $z_1 \in S_{xy_1} \cup H_{y_1 x}$. Consider the point z' satisfying $z_1 \in kz'$ and $\|k - z'\| = \|k - x\|$. Then project z' orthogonally onto kx and obtain a point z''. Consider a point y_2 between z'' and x on kx, and proceed with y_2 and x exactly as before with y_1 and x; we obtain a point $z_2 \in K \cap W_{xy_2}$. Since W_{xy_2} is disjoint from kz', but no point of K is separated on l_{kz_1} from k by z', it follows that $z_2 \notin l_{kz_1}$.

By iterating this procedure, we obtain a sequence of points $\{z_n\}_{n=1}^{\infty}$ such that the line-segments kz_n pairwise meet only at k, and $\|k - z_n\| > \|k - y_1\|$ for all n. Thus, $\sum_{n=1}^{\infty} \|k - z_n\| = \infty$, and $\mu_1(K) = \infty$. $\square$

Corollary 3. *If $K \in \mathcal{S}$ is a finite union of line-segments, then K is not rt-convex.*



(a) (b) (c)

Figure 4:

This does not mean that $\ker K$ cannot consist of a single point if K is rt-convex. For one of the simplest starshaped sets which are rt-convex, see Fig. 4(a). This example also shows that the Hausdorff dimension of a compact starshaped *rt*-convex set can be 1. Obviously, it can also be larger than 1, see Fig. 4(b).

The condition $\dim \ker K > 0$ does not guarantee the *rt*-convexity of K, see Fig. 4(c). However, higher dimension of the kernel "almost" does.

Theorem 9. *If* $K \in \mathcal{S}$ *and* $\dim \ker K \geq 2$, *then* K *is almost rt-convex.*

Proof. This is an immediate consequence of Theorem 1, because K is 2-connected and polygonally connected. The latter is obvious, for any dimension of $\ker K$.

Also the 2-connectedness is quickly verified. Take two arbitrary points $x, y \in K$. Since $\ker K$ is not included in l_{xy}, we can choose $k \in (\ker K) \setminus l_{xy}$. Let Π be the 2-plane containing x, y, k. If $\ker K$ is not included in Π, then for any choice of $k' \in (\ker K) \setminus \Pi$ the two polygonal paths xky and $xk'y$ lie in K and meet only at x, y. If $\ker K \subset \Pi$, consider a first point $k' \in \operatorname{relint} \ker K$ and a second point $k'' \in \operatorname{relint}(\ker K \cap \operatorname{conv}\{k', x, y\})$. For these two points it is again true that $xk'y \cup xk''y \subset K$ and $xk'y \cap xk''y = \{x, y\}$. $\square$

Theorem 10. *Let* $K \in \mathcal{S}$ *be different from a line-segment, but have* $\dim \ker K = 1$, *and let* L *be the line including* $\ker K$. *If* $K \cap L \subset p_L(K \setminus L)$, *then* K *is almost rt-convex.*

Proof. Put $\ker K = ab$. Let $K' = (K \setminus L) \cup ab$. This set is 2-connected. Indeed, let $x, y \in K'$. Since K is not a line-segment, there exists a point $z \notin L$.

If $x, y \in ab$, then they can be joined by two suitable broken lines inside $\operatorname{conv}\{a, b, z\} \subset K$, for example xzy and xgy, where g is the baricentre of $\operatorname{conv}\{x, y, z\}$.

Soppose now $x \notin ab$. If $xay \cap xby = \{x, y\}$, we are done. If not, then $xa \cap yb \neq \emptyset$ or $xb \cap ya \neq \emptyset$. Assume w.l.o.g. that the latter holds, and $\{c\} = xb \cap ya$. Then xcy and $xc'y$ are suitable broken lines, where $c' = (a + b)/2$.

By Theorem 1, K' is almost *rt*-convex, and the only pair of points in K' possibly without the *rt*-property is diametral. To finish the proof it remains to show that every pair of points $x \in K$, $y \in K \setminus K'$ enjoy the *rt*-convex.

We have $y \in K \cap L$. If $x \in K'$ and $y \neq p_L(x)$, then $\angle x p_L(x) y = \pi/2$. If $x \in K'$ and $y = p_L(x)$, then $\angle xya = \pi/2$. If $x \in K \setminus K'$, then $x = p_L(z)$ for some $z \in K \setminus L$ since $K \setminus K' \subset K \cap L$, whence $\angle zxy = \pi/2$. $\square$

While sets in $\mathcal{S}$ with higher kernel dimension are almost *rt*-convex, we saw that for single-point kernels, *rt*-convexity is rather the exception than the rule.

From the point of view of Baire categories, perhaps counter-intuitively, most sets in $\mathcal{S}$ have single-point kernels, as the second author proved in [5] (see the Corollary to Theorem 1 there). So, the question whether most sets in $\mathcal{S}$ are *rt*-convex or at least almost *rt*-convex becomes interesting.

Lemma 3. *For most $K \in \mathcal{S}$, we have* $\operatorname{card}(K \cap S_K) = d + 1$.

This is the version for starshaped sets of Lemma 1. The latter is a result from [4] which deals there with convex curves and surfaces. Its proof, given in [4] only for $d = 2$, can easily be modified to work for higher d, and adapted for the space $\mathcal{S}$, as well as for other Baire spaces of compact sets.

Lemma 4. *Most $K \in \mathcal{S}$ look at least half-dense from any of their points.*

This lemma is Theorem 1 in [7] for the space of starshaped sets, one of the spaces treated there.

Theorem 11. *Most $K \in \mathcal{S}$ are rt-convex.*

Proof. We shall restrict $\mathcal{S}$ to those compact starshaped sets K with $\mathbf{0} \in \ker K$. All known results about $\mathcal{S}$ remain of course true with this restriction. So, for example, for most $K \in \mathcal{S}$, $\ker K = \{\mathbf{0}\}$, and this will be assumed from now on.

Let $x, y \in K \setminus \{\mathbf{0}\}$. We verify the *rt*-property for these two points. Indeed, suppose on the contrary $K \cap W_{xy} = \emptyset$.

If $K \subset \operatorname{conv} S_{xy}$, then $K \cap S_{xy} = \{x, y\}$. But in this case S_{xy} is the circumsphere of K and $\operatorname{card}(K \cap S_{xy}) = 2$, contradicting Lemma 3.

Thus, $K \setminus \operatorname{conv} S_{xy} \neq \emptyset$.

Assume now that $\mathbf{0} \in \operatorname{conv} S_{xy}$. Choose $z \in K \setminus \operatorname{conv} S_{xy}$. Then $\mathbf{0}z \cap \overline{W_{xy}} \subset \{x, y\}$, so either $\mathbf{0}z$ cuts H_{xy} in x or $\mathbf{0}z$ cuts H_{yx} in y. Assume w.l.o.g. the first case. By Lemma 4, there are points in $K \setminus l_{\mathbf{0}z}$ arbitrarily close to z. For such a point z', $\mathbf{0}z'$ cuts H_{xy} in a point different from x, and a contradiction is obtained.

The remaining possibility is that $\mathbf{0} \notin \operatorname{conv} W_{xy}$. Suppose w.l.o.g. that H_{xy} separates $\mathbf{0}$ from y. Then, clearly, $x \in \mathbf{0}y$. By Lemma 4, there are points in $K \setminus l_{\mathbf{0}y}$ arbitrarily close to y. For such a point y', $\mathbf{0}y'$ cuts H_{xy} in a point different from x, and a contradiction is obtained again.

Now let $z \in K$. We verify the *rt*-property for $\mathbf{0}$ and z.

Let $\mathcal{S}_n$ be the set of all $K \in \mathcal{S}$ possessing a point $x \in \operatorname{ex} K$ with $\|x\| \geq 1/n$ such that $S_{\mathbf{0}x} \cap K = \{\mathbf{0}, x\}$. We show that $\mathcal{S}_n$ is nowhere dense.

Let $K \in \mathcal{S}$ and approximate K by $F \in \mathcal{S}$ with $\operatorname{ex} F$ finite. For each $x \in \operatorname{ex} F$ with $\|x\| \geq 1/n$, add a line-segment $\mathbf{0}y$ to F such that $\|x\| = \|x - y\|$ and $\|y\|$ is as small as desired. Obtain in this manner $F' \in \mathcal{S} \setminus \mathcal{S}_n$ close to F. Note that $S_{\mathbf{0}x}$ cuts $\mathbf{0}y$ at its midpoint. So, there exists a neighbourhood $\mathcal{V}$ of F' in $\mathcal{S}$ such that $\mathcal{V} \cap \mathcal{S}_n = \emptyset$.

Thus, $\mathcal{S}_n$ is nowhere dense, which implies that $\bigcup_{n=1}^{\infty} \mathcal{S}_n$ is of first category.

Hence, for most $K \in \mathcal{S}$, $S_{\mathbf{0}x} \cap K \neq \{\mathbf{0}, x\}$ for all $x \in \operatorname{ex} K$. Note that, in these sets K, $\mathbf{0}$ and z have the *rt*-property as soon as $z \in \mathbf{0}x$ for some $x \in \operatorname{ex} K$ (because, writing $z = \lambda x$, $S_{\mathbf{0}z} \cap K \supset S_{\mathbf{0}z} \cap \lambda K = \lambda(S_{\mathbf{0}x} \cap K)$), and this happens for every $z \in K \setminus \{\mathbf{0}\}$. □

Can the case of starshaped sets with higher-dimensional kernels be treated using Baire categories? Yes, it is enough to restrict $\mathcal{S}$ to the subspace $\mathcal{S}_C$ of those starshaped sets whose kernels include a given compact convex set C with $1 \leq \dim C \leq d$. We know, by Theorem 9, that all members of $\mathcal{S}_C$ are almost *rt*-convex if $\dim C \geq 2$; if $\dim C = 1$, even that is not guaranteed.

We take C in a linear $(\dim C)$-dimensional subspace L_C of $\mathbb{R}^d$.

We need the following old result (see the last four lines of [6]).

Lemma 5. *For most $K \in \mathcal{S}_C$, no point of $L_C \setminus C$ belongs to K.*

In fact, much more is known about most $K \in \mathcal{S}_C$ (see [6]); for example, that each $(\dim C + 1)$-dimensional subspace containing L_C and meeting $K \setminus L_C$ intersects K along a topological disc, which intersects L_C along C.

Theorem 12. *For any convex set C, most $K \in \mathcal{S}_C$ are rt-convex.*

Proof. Assume first $\dim C \geq 2$. Then $\dim \ker K \geq 2$, too. In the proof of Theorem 10 it is shown that each $K \in \mathcal{S}_C$ is 2-connected and polygonally connected. By Theorem 1, K is almost *rt*-convex, the exceptional pair, if it exists, being diametral. Lemma 3 remains true when restricted to $\mathcal{S}_C$, see the comment following Lemma 3. Thus, for most $K \in \mathcal{S}_C$,

$$\operatorname{card}(K \cap S_K) = d + 1 \geq 3,$$

which excludes the existence of an exceptional diametral pair. Hence most K are *rt*-convex.

Now assume $\dim C = 1$. If we can prove that $\dim K = 1$ and that the condition of Theorem 10 involving the line $L \supset \ker K$ is verified by most K, then the rest follows like in the just discussed case $\dim C \geq 2$. And indeed, both $\dim \ker K = 1$ and the mentioned condition are verified, since, for most K, $L \cap K = \ker K$, by Lemma 5. □

rt-convexity of finite sets.

Concerning the family $\mathcal{E}$ of all finite sets in $\mathbb{R}^2$, in order to obtain classes of *rt*-convex sets in $\mathcal{E}$, finding some sufficient conditions for *rt*-convexity (or almost *rt*-convexity) would be helpful. However, here we don't enjoy the support of Corollary 1 or Theorems 1, 2, 3.

A few examples of finite *rt*-convex sets can easily be constructed. See the next sections for a classification of all 4-point and 5-point *rt*-convex sets.

In order to produce large families of *rt*-convex sets in $\mathcal{E}$, one can use the next straightforward but useful fact, valid in arbitrary dimension, and also for infinite sets.

Theorem 13. *If $M \subset \mathbb{R}^d$ is rt-convex and L is an affine subspace of $\mathbb{R}^d$, then $M \cup p_L(M)$ is also rt-convex.*

Of course, starting with any *rt*-convex set one can produce in this way infinitely many new *rt*-convex sets, each of them including the preceding one.

Let $\mathcal{R}$ be the family of all triples $\{x, y, z\}$ such that $\operatorname{conv}\{x, y, z\}$ is a non-degenerate right triangle, i.e. the family $\mathcal{F}'$ from the Introduction.

Consider $\{a, b, c\} \in \mathcal{R}$ with $\angle abc = \frac{\pi}{2}$, and d_0 the orthogonal projection of b on ac. Let d_{2k+1} denote the orthogonal projection of d_{2k} on ab, and d_{2k+2} the orthogonal projection of d_{2k+1} on d_0a, where $k = 0, 1, 2, ...$ We call the sequence of points $b, d_0, d_1, d_2, ..., d_n$ an *n-zigzag* in the acute angle a. An *even* (*odd*) zigzag is an n-zigzag such that n is even (odd). Infinite zigzags can also be conceived.

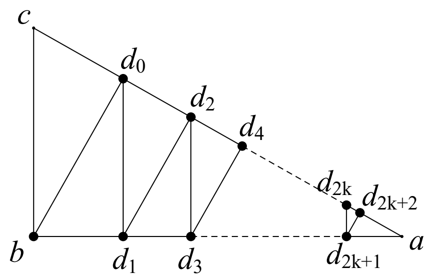


Figure 5:

Let $\mathcal{L}$ and $\mathcal{H}$ be two families of parallel lines in $\mathbb{R}^2$ such that l and h are perpendicular, for any $l \in \mathcal{L}$ and $h \in \mathcal{H}$. Let $P = \{p \in l \cap h : l \in \mathcal{L}, h \in \mathcal{H}\}$. For $p \in P$, let l_p denote the line in $\mathcal{L}$ containing p, and put $l_p^* = l_p \setminus \{p\}$. Introduce h_p and h_p^* analogously. A set $S \subset P$ is called *lattice-like*. Consider the conditions:

(i) $l_a = l_b \Rightarrow (h_a^* \cup h_b^*) \cap S \neq \emptyset$,

(ii) $h_a = h_b \Rightarrow (l_a^* \cup l_b^*) \cap S \neq \emptyset$,

(iii) $l_a \neq l_b$ and $h_a \neq h_b \Rightarrow (l_a \cap h_b) \cup (h_a \cap h_b) \neq \emptyset$.

Remark. The following sets are *rt*-convex:

(a) a set consisting of pairs of antipodal points of a circle and at most one other point on the circle;

(b) a lattice-like set verifying (i)–(iii);

(c) a set containing a vertex at an acute angle of a right triangle and a zigzag in the other acute angle, where the second endpoint of the hypothenuse may be included or not;

(d) a set containing two even (or two odd) zigzags of a right triangle in its two acute angles, where each endpoint of the hypothenuse may be included or not.

Note that all sets mentioned at (c), (d) can be obtained by repeatedly using Theorem 13.

Theorems 14 and 15 present two different sufficient conditions for a finite lattice-like set to be *rt*-convex. They correspond to two classes of *rt*-convex sets in $\mathbb{Z}^2$ (called here positive sets without peaks, respectively convex sets).

Put $[m,n] = \{m, m+1, ..., n-1, n\}$, where $m,n \in \mathbb{Z}$ and $m < n$.

Let the families $\mathcal{L}$ and $\mathcal{H}$ be finite, and $\operatorname{card}\mathcal{L} = u$, $\operatorname{card}\mathcal{H} = v$.

By numbering the families $\mathcal{L}$ and $\mathcal{H}$ with integers, every point $l \cap h$ becomes a pair $(x,y) \in \mathbb{Z}^2$, P becomes $[1,u] \times [1,v]$, and $S \subset P$ becomes a subset M_S of $[1,u] \times [1,v]$.

A function $f : [1,u] \to \mathbb{Z}$ is called here *positive* if $f \geq 0$, and $f(x) > 0$ for all $x \in [1,u]$ with at most one exception.

A set $M \subset \mathbb{Z}^2$ is called *positive* if, for some positive function $f : [1,u] \to \mathbb{Z}$,

$$M = \{(x,y) : 1 \leq x \leq u \text{ and } 0 \leq y \leq f(x)\}.$$

We say that M has a *peak* at $(x, f(x)) \in M$ if f has an absolute maximum at $x \in [1,u]$ and $f(x) \geq f(x') + 2$ for all $x' \neq x$. A peak is a *superpeak* if strict inequality holds.

Theorem 14. *If M_S is positive and has no superpeaks, then S is almost rt-convex. If M_S has no peaks, then S is rt-convex.*

Proof. Let $M \subset \mathbb{Z}^2$ be positive. Let $(x,y),(x',y') \in M$ and assume first that $x \neq x'$ and $y < y'$. Then $(x',y) \in M$ sees the pair $\{(x,y),(x',y')\}$ under a right angle. If $y > y'$, then $(x,y') \in M$ sees the pair $\{(x,y),(x',y')\}$ under a right angle. If $y = y' > 0$, then (x,y) sees the pair $\{(x,0),(x',y)\}$ under a right angle. If $y = y' = 0$, we remember that $f(x)$ or $f(x')$, say $f(x)$, is non-zero, f being positive. Then $(x,0)$ sees the pair $\{(x,1),(x',0)\}$ under a right angle.

Assume now that $x = x'$, and also w.l.o.g. that $y > y'$. If f has no maximum at x, or not only at x, then it has a maximum at $x'' \neq x$ and $f(x'') \geq f(x)$. Then (x,y) sees the pair $\{(x,y'),(x'',y)\}$ under a right angle.

Assume now that x is the only maximum of f. If M has no peak, then $f(x'') = f(x) - 1$ for some $x'' \neq x$. Hence $y' \leq f(x'')$ and $(x'',y') \in M$. The point (x,y') sees the pair $\{(x,y),(x'',y')\}$ under a right angle.

Hence M is almost *rt*-convex in case of absence of superpeaks. If M has the peak $(x, f(x))$, the exceptional pair is $\{(x,f(x)),(x,f(x)-1)\}$. If M has no peaks, it is *rt*-convex.

Now, observe that all triples in $\mathcal{R}$ with points in M correspond to triples in $\mathcal{R}$ with points in S, if $M = M_S$. $\square$

Let now $f : [1,u] \to \mathbb{Z}$ be unimodal, i.e. non-decreasing on a subinterval $[1,w]$ and non-increasing on $[w,u]$, where $1 \leq w < u$, and let $g : [1,w] \to \mathbb{Z}$ be non-increasing, such that $g(1) \leq \min\{f(1), f(u)\}$ if $f(1) \neq f(u)$, and $g(1) < f(1)$

otherwise. Define $h : [w+1, u] \to \mathbb{Z}$ as follows: Let $h(w+1) = g(w)$. For $w+2 \leq i \leq u$, let $h(i) = g(x(i))$, where $x(i) = \min\{\xi \leq w : f(\xi) \geq f(i)\}$.

It is easily checked that, defined in this way, h is non-decreasing.

Indeed, let $z, z' \in [w+1, u]$, with $z < z'$. Then $f(z) \geq f(z')$, which implies $x(z) \geq x(z')$, whence $g(x(z)) \leq g(x(z'))$, i.e. $h(z) \leq h(z')$.

A set $M \subset \mathbb{Z}^2$ will be called *convex* if, for some functions f, g, h defined as above,

$$M = \{(x, y) : 1 \leq x \leq w \text{ and } g(x) \leq y \leq f(x)\} \\ \cup \{(x, y) : w < x \leq u \text{ and } h(x) \leq y \leq f(x)\}.$$

Theorem 15. *If M_S is convex, then S is rt-convex.*

Proof. Again, we show that M_S itself is *rt*-convex.

Suppose that $(i, j), (i', j') \in M_S$ and $i' < i$, $j' < j$. If $(i', j) \in M_S$, we are done. If $(i', j) \notin M_S$ and $x^* = \min\{\xi \leq w : f(\xi) \geq j\}$, then $i' < x^*$. It follows that $g(i') \geq g(x^*)$, while $f(i') < f(x^*)$.

Let $i^* = \max\{\xi : f(\xi) \geq j\}$. Clearly, $i^* \geq i$ and $f(i^*) \geq j$. Hence, $x(i^*) \geq x^*$, yielding $h(i^*) = g(x(i^*)) \leq g(x^*) \leq g(i') \leq j'$.

Since h is non-decreasing, $h(i) \leq h(i^*) \leq j'$. Hence, $(i, j') \in M_S$ and (i, j), (i', j') have the *rt*-property.

Suppose now that $(i, j), (i', j') \in M_S$ and $i' < i$, $j' > j$. If $f(i) \geq j'$ then $(i, j') \in M_S$ and we are done. Otherwise, $f(i) < j'$ and $f(i') \geq j'$ imply $x(i) \leq i'$. This in turn yields $g(x(i)) \geq g(i')$, that is $h(i) \geq g(i')$. A fortiori $j \geq g(i')$, whence $(i', j) \in M_S$.

For $(i, j), (i', j) \in M_S$, and $i > i'$, we have $\{(i, j), (i', j), (i', j-1)\}) \in \mathcal{R}$ if $j > g(1)$, and $\{(i, j), (i', j), (i', j+1)\}) \in \mathcal{R}$ if $j \leq g(1)$.

For $(i, j'), (i, j) \in M_S$, and $j > j'$, we have $\{(i, j), (i, j'), (i+1, j')\}) \in \mathcal{R}$ if $i \leq w$, and $\{(i, j), (i, j'), (i-1, j')\}) \in \mathcal{R}$ otherwise. $\square$

Classification of 4-point *rt*-convex sets in the plane

In this and the next two sections we provide a classification of all 4-point and 5-point *rt*-convex sets in $\mathbb{R}^2$.

Theorem 16. *There are precisely three different types of 4-point rt-convex sets in the plane:*

type 4-1*: the three vertices of a right triangle and the foot of the height at the right angle;*

type 4-2*: the four vertices of a rectangle;*

type 4-3*: the vertex of an acute angle of a right triangle together with the* 1*-zigzag in the other acute angle.*

For an illustration of Theorem 16, see Figure 6.

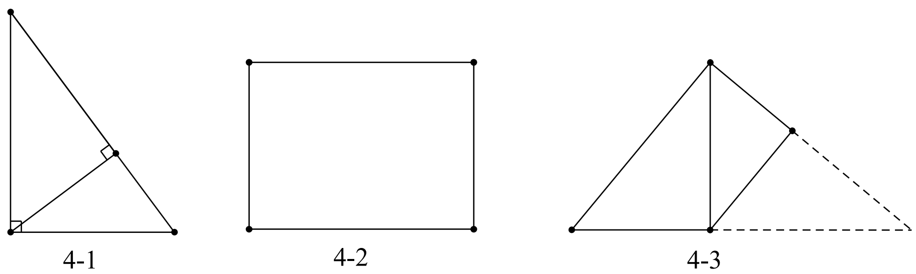
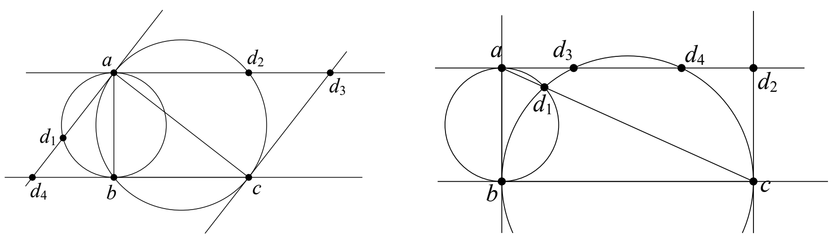


Figure 6: 4-point rt-convex sets

Proof. Let $T = \{a, b, c, d\}$ be a 4-point rt-convex set. By the definition, for any two points, say, $a, b \in T$, there is a point, say $c \in T$ such that $\{a, b, c\} \in \mathcal{R}$. We assume without loss of generality that $\angle abc = \frac{\pi}{2}$. Now we consider the position of d. Since for any $x \in \{a, b, c\}$, there exists $y \in \{a, b, c\} \setminus \{x\}$ such that $\{d, x, y\} \in \mathcal{R}$, there are three cases to consider.



(a) $d \in K_{ab} \cap K_{ac}$. (b) $d \in K_{ab} \cap K_{bc}$.

Figure 7: 4-point rt-convex sets.

Case 1. $\{d, a, b\} \in \mathcal{R}$, $\{d, a, c\} \in \mathcal{R}$. See Figure 7 (a). Clearly $d \notin S_{ab} \cap S_{ac}$. Thus, the possible positions for d are $d_1 \in S_{ab} \cap H_{ac} \setminus \{a\}$ and in this case T is of type 4-3; $d_2 \in H_{ab} \cap S_{ac} \setminus \{a\}$ and T is of type 4-2; $d_3 \in H_{ab} \cap H_{ca}$ and T is again of type 4-3; $d_4 \in H_{ba} \cap H_{ac}$ and then T is of type 4-1.

Case 2. $\{d, a, b\} \in \mathcal{R}$, $\{d, c, b\} \in \mathcal{R}$. See Figure 7 (b). Clearly $d \notin l_{ab}$, $d \notin l_{bc}$. So, the solutions for d are $d_1 \in S_{ab} \cap S_{bc} \setminus \{b\}$, and then T is of type 4-1; $d_2 \in H_{ab} \cap H_{cb}$ and T is of type 4-2; $\{d_3, d_4\} = H_{ab} \cap S_{cb}$ (under the condition $\|b - c\| \geq 2\|a - b\|$), when T is of type 4-3.

Case 3. $\{d, b, c\} \in \mathcal{R}$, $\{d, a, c\} \in \mathcal{R}$. Then T is as described in Case 1, to which Case 3 is symmetrical.

We shall call the three 4-point rt-convex sets described above the *4-point rt-convex sets generated by a, b, c*.

Classification of planar 5-point rt-convex sets with 4-point rt-convex subsets

We are going to classify all possible configurations for a 5-point rt-convex set $F = \{a, b, c, d, e\}$.

In this section we consider the case that F contains a 4-point rt-convex subset, which can be w.l.o.g. supposed to be $F_1 = \{a, b, c, d\}$, generated by a, b, c, where $\angle abc = \pi/2$.

By the definition of rt-convexity, e must meet one of the following 7 conditions.

C1. $\{\{e, a, b\},\ \{e, c, d\}\} \subset \mathcal{R}$;
C2. $\{\{e, a, c\},\ \{e, b, d\}\} \subset \mathcal{R}$;
C3. $\{\{e, a, d\},\ \{e, b, c\}\} \subset \mathcal{R}$;
C4. $\{\{e, a, b\},\ \{e, a, c\},\ \{e, a, d\}\} \subset \mathcal{R}$;
C5. $\{\{e, b, a\},\ \{e, b, c\},\ \{e, b, d\}\} \subset \mathcal{R}$;
C6. $\{\{e, c, a\},\ \{e, c, b\},\ \{e, c, d\}\} \subset \mathcal{R}$;
C7. $\{\{e, d, a\},\ \{e, d, b\},\ \{e, d, c\}\} \subset \mathcal{R}$.

According to the different configurations of F_1, there are three cases to consider.

Case 1. F_1 is of type 4-1. Then d is the orthogonal projection of b on ac.

Clearly, C1 and C3 generate the same F, and so do C4 and C6.

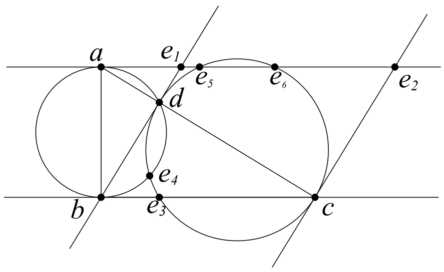


Figure 8: F_1 is of type 4-1, $e \in W_{ab} \cap W_{cd}$

If e satisfies C1, then $e \in W_{ab} \cap W_{cd}$. All six possible positions $e_1, ..., e_6$ of e are shown in Figure 8, where $\{e_1\} = H_{ab} \cap H_{dc}$, $\{e_2\} = H_{ab} \cap H_{cd}$, $\{e_3\} = H_{ba} \cap S_{cd} \setminus \{c\}$, $\{e_4\} = S_{ab} \cap S_{cd} \setminus \{d\}$, and $\{e_5, e_6\} = H_{ab} \cap S_{cd}$. The latter intersection is non-empty if and only if

$$0 < \sin \angle acb \leq x_0,$$

where x_0 is the real root of $x^3 + x^2 + x - 1 = 0$. In case of equality e_5 and e_6 exist, but coincide.

Indeed, let λ denote the distance from the centre of S_{cd} to the line H_{ab}. Then $\lambda - \frac{1}{2}\|c - d\| = \|a - b\| - \frac{1}{2}\|c - d\| \sin \angle acb - \frac{1}{2}\|c - d\| = \frac{1}{2}(2\|a - b\| - \|b - c\| \cos \angle acb(\sin \angle acb + 1) = \frac{1}{2}\|a - b\|(2 - \frac{\cos^2 \angle acb(\sin \angle acb+1)}{\sin \angle acb}) \leq 0$, i.e. $\cos^2 \angle acb(\sin \angle acb+1) \geq 2 \sin \angle acb$. Putting $x = \sin \angle acb$, this means $x^3+x^2+x-1 \leq$

0. The left side is monotone increasing everywhere, so the inequality holds on the interval $]0, x_0]$.

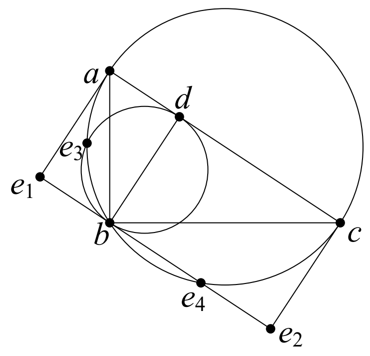


Figure 9: F_1 is of type 4-1, $e \in W_{ac} \cap W_{bd}$

If e satisfies C2, then $e \in W_{ac} \cap W_{bd}$. Clearly $H_{db} \cap W_{ac} = \emptyset$. We assume without loss of generality that $\|a - b\| \leq \|b - c\|$. Then we have for e the solutions $e_1 \in H_{ac} \cap H_{bd}$, $e_2 \in H_{ca} \cap H_{bd}$, $e_3 \in S_{ac} \cap S_{bd} \setminus \{b\}$, and $e_4 \in S_{ac} \cap H_{bd} \setminus \{b\}$. Note that e_3 and e_4 exist under the condition $\|a - b\| < \|b - c\|$. See Figure 9.

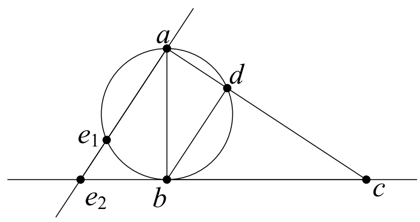


Figure 10: F_1 is of type 4-1, $e \in W_{ab} \cap W_{ac} \cap W_{ad}$

If e satisfies C4, then $e \in W_{ab} \cap W_{ac} \cap W_{ad}$. Clearly $W_{ac} \cap W_{ad} = H_{ad} \setminus \{a\}$. Since $H_{ab} \cap H_{ad} \setminus \{a\} = \emptyset$, we obtain 2 possible configurations for F as shown in Figure 10, where $e_1 \in H_{ad} \cap S_{ab} \setminus \{a\}$, and $e_2 \in H_{ad} \cap H_{ba}$.

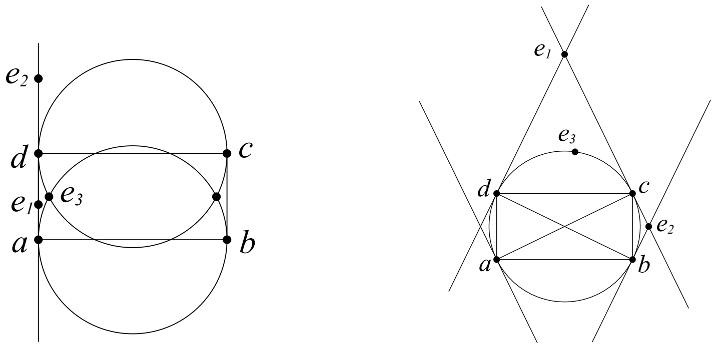
Since $W_{ba} \cap W_{bc} = (H_{cb} \cap H_{ab}) \cup \{d\}$, which has an empty intersection with W_{bd}, there is no point e satisfying C5. Similarly, $W_{da} \cap W_{db} \cap W_{dc} = \emptyset$, and there is no point e satisfying C7.

Case 2. F_1 is of type 4-2. Then C1, C3 generate the same set F, and so do C4, C5, C6 and C7.

If e satisfies C1, then $e \in W_{ab} \cap W_{cd}$. Clearly, F can be of 3 different kinds, as shown in Figure 11 (a), where $e_1 \in ad \setminus \{a, d\}$, $e_2 \in H_{ab} \setminus ad$, and $e_3 \in S_{ab} \cap S_{cd}$, supposing w.l.o.g. for the latter that $\|a - b\| \geq \|a - d\|$.

Note that the solution provided here by e_2 includes the previous ones obtained for F_1 of type 4-1, case C2, both e_1, e_2, and case C4, e_1.

If e satisfies C2, then $e \in W_{ac} \cap W_{bd}$. Clearly, $S_{ac} = S_{bd}$. Then F can again be of 3 different kinds, as shown in Figure 11 (b), where $\{e_1\} = H_{ca} \cap H_{db}$, $\{e_2\} = H_{ca} \cap H_{bd}$, and $e_3 \in S_{ac} \setminus \{a, b, c, d\}$.



(a) (b)

Figure 11: F_1 is of type 4-2

Furthermore, $W_{ab} \cap W_{ad} \setminus \{c\} = S_{ab} \cap S_{ad} \setminus \{a\} \in \text{int conv} S_{ac})$, which means that $W_{ab} \cap W_{ac} \cap W_{ad} = \emptyset$. So there is no point e satisfying C4; consequently, C5, C6, C7 cannot be satisfied either.

Case 3. F_1 is of type 4-3.

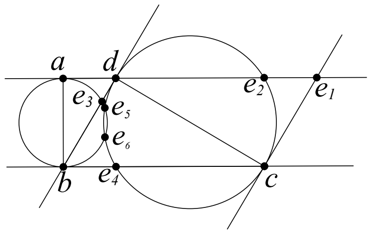


Figure 12: F_1 is of type 4-3, $e \in W_{ab} \cap W_{cd}$

If e satisfies C1, then $e \in W_{ab} \cap W_{cd}$. Thus we have as solutions $\{e_1\} = H_{ab} \cap H_{cd}$, $\{e_2\} = H_{ab} \cap S_{cd} \setminus \{d\}$, $\{e_3\} = S_{ab} \cap H_{dc} \setminus \{b\}$, $\{e_4\} = H_{ba} \cap S_{cd} \setminus \{c\}$, and $\{e_5, e_6\} = S_{ab} \cap S_{cd}$, the latter under the condition $\|a - d\| + \|b - c\| \leq \|a - b\| + \|c - d\|$, see Figure 12. In case of equality, $e_5 = e_6$.

Note that e_2, e_4 don't lead to new solutions, being particular cases of F obtained for F_1 of type 4-2, C1, e_1, e_2, respectively.

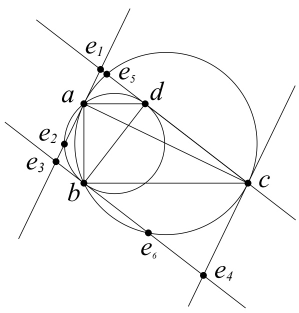


Figure 13: F_1 is of type 4-3, $e \in W_{ac} \cap W_{bd}$

If e satisfies C2, then $e \in W_{ac} \cap W_{bd}$. This leads to 6 different configurations shown in Figure 13, where $e_1 \in H_{ac} \cap H_{db}$, $e_2 \in H_{ac} \cap S_{bd} \setminus \{a\}$, $e_3 \in H_{ac} \cap H_{bd}$, $e_4 \in H_{ca} \cap H_{bd}$, $e_5 \in S_{ac} \cap H_{db} \setminus \{c\}$, $e_6 \in S_{ac} \cap H_{bd} \setminus \{b\}$.

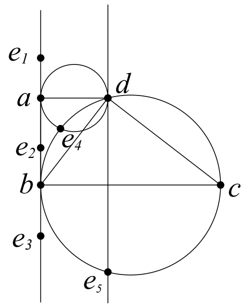


Figure 14: F_1 is of type 4-3, $e \in W_{ad} \cap W_{bc}$

If e satisfies C3, then $e \in W_{ad} \cap W_{bc}$. Clearly, $H_{ad} = H_{bc}$. So we have $e \in H_{ad} \setminus \{a, b\}$, which leads to 3 different types of F, see e_1, e_2, e_3 in Figure 14, or the solutions $e_4 \in S_{ad} \cap S_{bc} \setminus \{d\}$, and $e_5 \in H_{da} \cap S_{bc} \setminus \{d\}$.

Clearly, $W_{ab} \cap W_{ad} \subset \operatorname{conv}\{a, b, c, d\}$, whereas $W_{ac} \cap \operatorname{conv}\{a, b, c, d\} \setminus \{a, b, c, d\} = \emptyset$. So there is no point e satisfying C4.

Let $\{p\} = H_{ab} \cap H_{cb}$. Then $W_{ba} \cap W_{bc} \subset \operatorname{conv}\{a, b, d\} \cup dp$, which has an empty intersection with W_{bd}. Thus, $W_{ba} \cap W_{bc} \cap W_{bd} = \emptyset$, which means that there is no point e satisfying C5.

If e satisfies C6, then $e \in W_{ca} \cap W_{cb} \cap W_{cd}$. It is not hard to check that F must be as shown in Figure 15, where $e \in S_{ca} \cap H_{cb} \cap S_{cd}$.

Note that this coincides with the solution e_2 if e satisfies C1, itself a particular case of a previous solution.

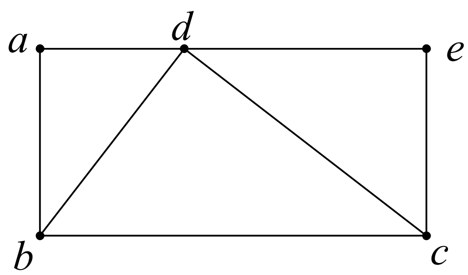


Figure 15: F_1 is of type 4-3, $e \in W_{ca} \cap W_{cb} \cap W_{cd}$.

If e satisfies C7, $e \in W_{da} \cap W_{db} \cap W_{dc}$. Then

$$W_{da} \cap W_{dc} = (S_{dc} \cap (S_{da} \cup H_{da}) \setminus \{d\}) \cup (H_{cd} \cap (H_{ad} \cup H_{da})),$$

which contains 4 points, see Figure 16 (a).

One of them, $\{e_1\} = H_{da} \cap H_{ba}$, lies on W_{db}, too, and is a solution, but not a new one, as it coincides with e_4 if e satisfies C1, which in turn is a particular

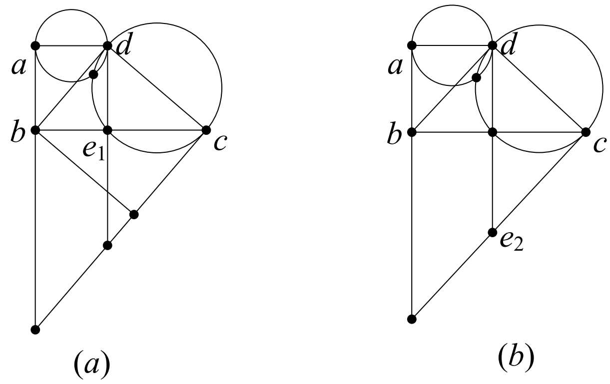


Figure 16: F_1 is of type 4-3, $e_1, e_2 \in W_{da} \cap W_{db} \cap W_{dc}$

case of another solution, as we noticed. A second one, $\{e_2\} = H_{cd} \cap H_{da}$, is a solution in case $\|a - b\| = \|a - d\|$, see Fig. 16 (b). This is also not new, being a particular case of the solution obtained for F_1 of type 4-2, C2, e_1.

Classification of planar 5-point rt-convex sets without 4-point rt-convex subsets

Now we consider the case when $F = \{a, b, c, d, e\}$ is a 5-point rt-convex set containing no 4-point rt-convex set. We still assume that $\{a, b, c\} \in \mathcal{R}$, but not any more $\angle abc = \pi/2$.

Since the analysis of all cases leading to our classification is lengthy, but does not offer any new kinds of approach, and always follow similar methods, we decide to present here rather sketchily some of the proofs, but all pertinent figures.

For the point $z \in \{d, e\}$, if there exist $x, y \in \{a, b, c\}$ such that $\{x, y, z\} \in \mathcal{R}$, then we say that z is *rt-good*.

There are two cases to be considered, first that both d and e are rt-good, and then that at least one of them is not rt-good.

Assume first that both d and e are rt-good.

Suppose that $x_1, y_1 \in \{a, b, c\}$ satisfy $\{d, x_1, y_1\} \in \mathcal{R}$, $x_2, y_2 \in \{a, b, c\}$ satisfy $\{e, x_2, y_2\} \in \mathcal{R}$.

Case 1. x_1y_1 and x_2y_2 are the same side of the right triangle abc.

We assume w.l.o.g. that $x_1y_1 = x_2y_2 = ab$. Since F does not contain a 4-point rt-convex set generated by $\{a, b, c\}$, we have $\{d, a, b\} \in \mathcal{R} \Longrightarrow \{d, a, c\} \notin \mathcal{R}, \{d, b, c\} \notin \mathcal{R}$, and therefore $\{d, c, e\} \in \mathcal{R}$; similarly, $\{e, a, b\} \in \mathcal{R} \Longrightarrow \{e, a, c\} \notin \mathcal{R}, \{e, b, c\} \notin \mathcal{R}$, and therefore $\{e, c, d\} \in \mathcal{R}$. Thus the points d, e satisfy

$$\{\{d, a, b\}, \{d, c, e\}, \{e, a, b\}\} \subset \mathcal{R}.$$

Recalling that $\{a, b, c\} \in \mathcal{R}$, we actually have $c, d, e \in W_{ab}$ satisfying $\{c, d, e\} \in \mathcal{R}$.

Subcase 1.1. At least two points from $\{c, d, e\}$, say c and d, lie in one of the sets H_{ab}, H_{ba}, S_{ab}.

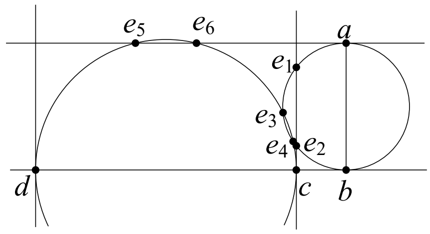


Figure 17: $c, d \in H_{ba}$, $c \in \operatorname{relint} bd$.

(I) $c, d \in H_{ba}$, $c \in \operatorname{relint} bd$. See Figure 17, where $e \in S_{ab} \cap H_{cd}$ if $2\|b-c\| \leq \|a-b\|$ (see e_1, e_2); or $e \in S_{ab} \cap S_{cd}$ if $\|a+b-c-d\| \leq \|a-b\| + \|c-d\|$ (see e_3, e_4); or $e \in H_{ab} \cap S_{cd}$ under the condition $\|c-d\| \geq 2\|a-b\|$ (see e_5, e_6).

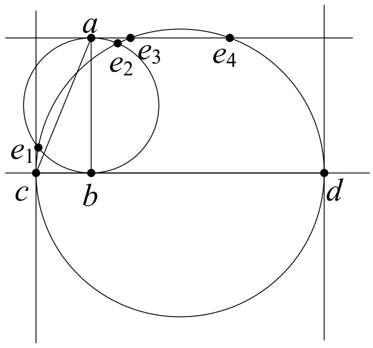


Figure 18: $c, d \in H_{ba}$, $b \in \operatorname{relint} cd$.

(II) $c, d \in H_{ba}$, $b \in \operatorname{relint} cd$, and we assume without loss of generality that $\|b-c\| \leq \|b-d\|$. See Figure 18, where e_3, e_4 appear only under the condition $\|c-d\| \geq 2\|a-b\|$.

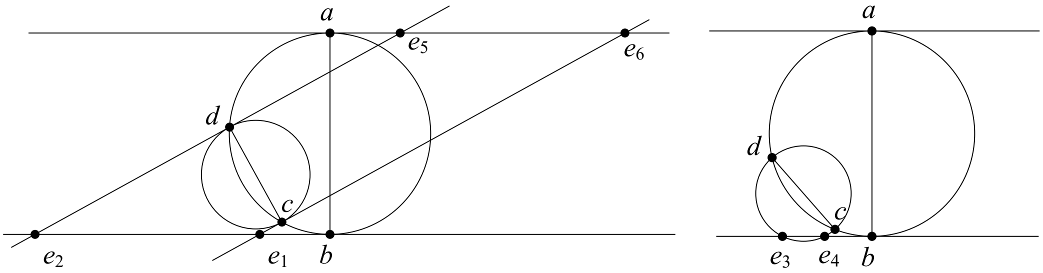
(III) $c, d \in S_{a,b}$. Then $\|c-d\| < \|a-b\|$ and cd is not parallel to ab. See Figure 19, where $e \in H_{ba} \cap (H_{cd} \cup H_{dc})$ (see e_1, e_2); or $e \in H_{ba} \cap S_{cd}$ if the distance from the point $(c+d)/2$ to the line H_{ba} is less than or equal to $\|c-d\|/2$ (see e_3, e_4); or $e \in H_{ab} \cap (H_{cd} \cup H_{dc})$ (see e_5, e_6). Note that $e \in S_{ab} \cap (H_{cd} \cup H_{dc})$ is not an option, because $\{a, b, c, e\}$ becomes *rt*-convex.

Subcase 1.2. Exactly one of the points c, d, e, belongs to each of the sets H_{ab}, H_{ba}, S_{ab}.

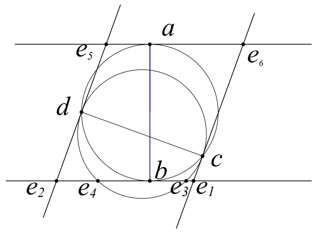
We suppose w.l.o.g. that $c \in H_{ba}$, $d \in H_{ab}$, $e \in S_{ab}$, and $\|a-d\| \leq \|b-c\|$.

If c, d lie on the same side of l_{ab}, then see Figure 20 (a) . If c, d lie on different sides of l_{ab}, then see Figure 20 (b) and (c) .

Case 2. x_1y_1 and x_2y_2 are two different sides of the right triangle abc. We assume without loss of generality that $x_1 = x_2 = b$, $y_1 = a$, $y_2 = c$.

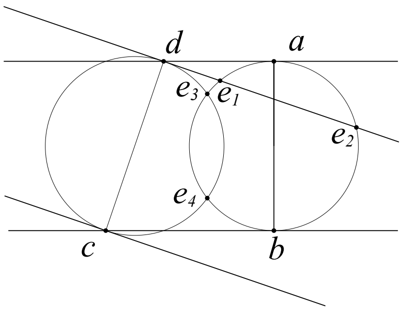


(a) c, d lie on a same semicircle bounded by ab.

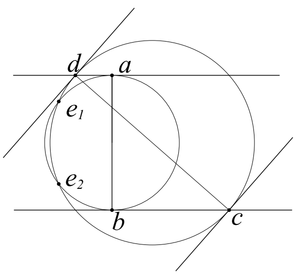


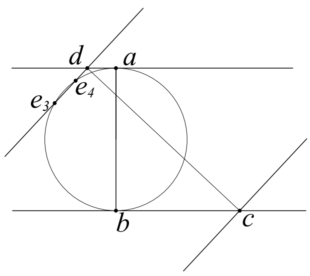
(b) c, d lie on different semicircles bounded by ab.

Figure 19: $c, d \in S_{a,b}$, $|cd| < |ab|$ and $cd \nparallel ab$.



(a)





(b) (c)

Figure 20: $c \in H_{ba}$, $d \in S_{ba}$, $e \in H_{ab}$.

Since F does not contain a 4-point rt-convex set generated by $\{a, b, c\}$, we have $\{d, a, b\} \in \mathcal{R} \Longrightarrow \{d, a, c\} \notin \mathcal{R}, \{d, b, c\} \notin \mathcal{R}$, and therefore $\{d, c, e\} \in \mathcal{R}$; similarly, $\{e, b, c\} \in \mathcal{R} \Longrightarrow \{e, a, b\} \notin \mathcal{R}, \{e, a, c\} \notin \mathcal{R}$, and therefore $\{e, a, d\} \in \mathcal{R}$. Thus the points d, e satisfy

$$\{\{d, a, b\}, \{d, c, e\}, \{e, b, c\}, \{e, a, d\}\} \subset \mathcal{R},$$

i.e., $d \in W_{ab}$, and $e \in W_{cd} \cap W_{bc} \cap W_{ad}$.

Subcase 2.1. $\angle abc = \frac{\pi}{2}$.

(I) $d \in H_{ba}$.

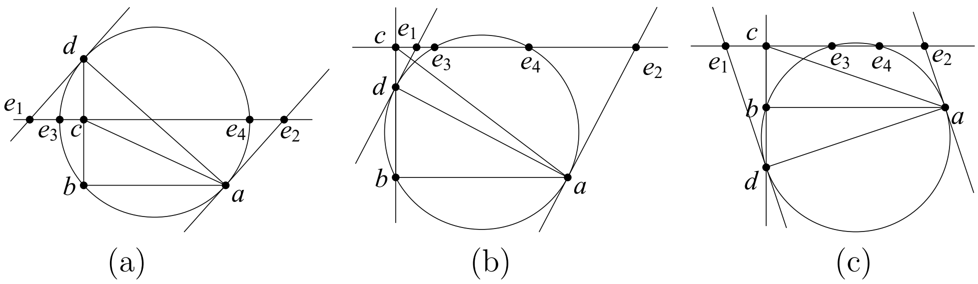


Figure 21: $\angle abc = \frac{\pi}{2}$, $d \in H_{ba}$

Then $W_{cd} \cap W_{bc} \subset H_{cb} = H_{cd}$, and therefore $e \in H_{cb} \cap W_{ad}$. If $c \in \operatorname{relint} bd$, see Figure 21 (a); if $d \in \operatorname{relint} bc$, then see Figure 21 (b); If $d \in \operatorname{relint} bc$, then see Figure 21 (c).

(II) $d \in H_{ab}$. See Figure 22.

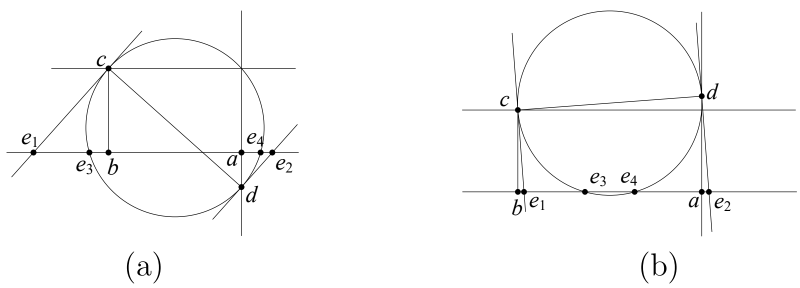


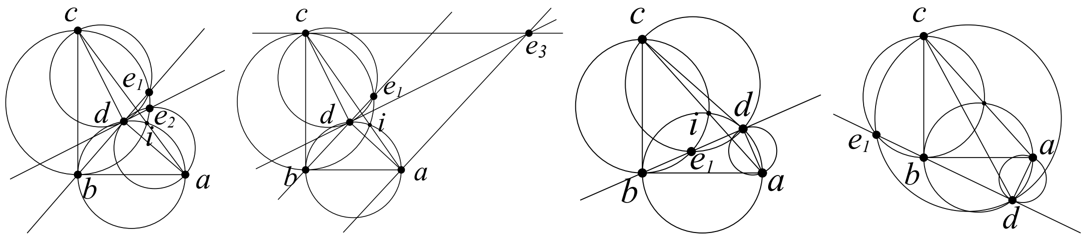
Figure 22: $\angle abc = \frac{\pi}{2}$, $d \in H_{ab}$

(III) $d \in S_{ab}$. See Figure 23.

Subcase 2.2. $\angle abc < \frac{\pi}{2}$.

We assume without loss of generality that $\angle bac = \pi/2$.

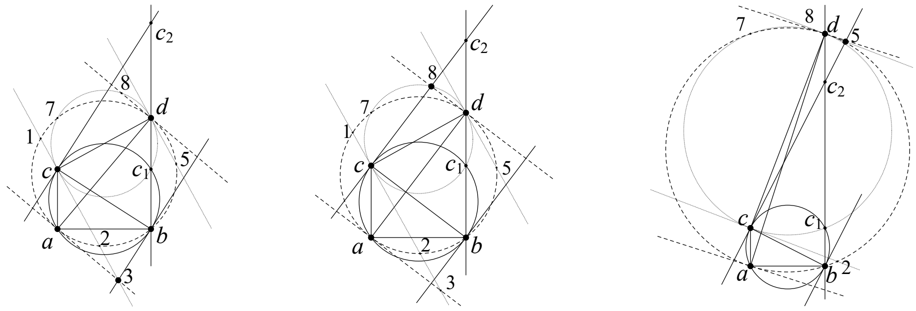
(I) $d \in H_{ab}$. If $d \in \operatorname{relint} ac$, we exchange the labels of a, b and c, d (relabel a by b, b by a, c by d, d by c). Then F admits the solutions described in Subcase 1.2.1 (I) under the condition $c \in \operatorname{relint} bd$, as shown in Figure 21 (a). If $c \in \operatorname{relint} ad$, we do the same relabeling and F has the solutions described in Subcase 1.2.1 (I) under the condition $d \in \operatorname{relint} bc$, as shown in Figure 21 (b).



(a) (b) (c) (d)

Figure 23: $\angle abc = \frac{\pi}{2}$, $d \in S_{ab}$

(II) $d \in H_{ba}$. Let c_1 be the orthogonal projection of c on the line H_{ba} and $\{c_2\} = H_{cb} \cap H_{ba}$.



(a) (b) (c)

Figure 24: $\angle abc < \frac{\pi}{2}$, $d \in H_{ba}$

If $d \in bc_1 \setminus \{b, c_1\}$, then there is a possible position of $e = 3$ as shown in Figure 24 (a). Further, if $\|a - c\| < \|a - b\|$, then there is a possible position of $e = 8$ as shown in Figure 24 (b).

If $d \in H_{ba}$ and $c_2 \in \operatorname{relint} c_1 b$, we have a position of $e = 5$ as shown in Figure 24 (c). If $d \in H_{ba}$ and $b \in \operatorname{relint} c_1 d$, then we have all the possible positions of e as shown in Figure 25 (a) to (f) respectively. Furthermore, if $\|a - c\| > \|a - b\|$, then we also have a position of e as shown in Figure 26.

(III) $d \in S_{ab}$. Let i be the orthogonal projection of a on the line segment bc. Clearly $d \neq i$.

(i) $d \in \widehat{ai}$, where $\widehat{ai}$ denotes the arc of S_{ab} from a to i in dextrorsum sense. If $\|a - c\| < \|a - b\|$, then see Figure 27 (a) and (b). If $\|a - c\| \geq \|a - b\|$, then see Figure 27 (c).

(ii) $d \in \widehat{ib}$. See Figure 28.

(iii) $d \in \widehat{ba}$. See Figure 29.

Suppose now that at least one of the points d, e is not rt-good.

We assume without loss of generality that d is not rt-good. By the definition, we must have $\{\{d, a, e\}, \{d, b, e\}, \{d, c, e\}\} \subset \mathcal{R}$. Recalling that $\{a, b, c\} \in \mathcal{R}$, we

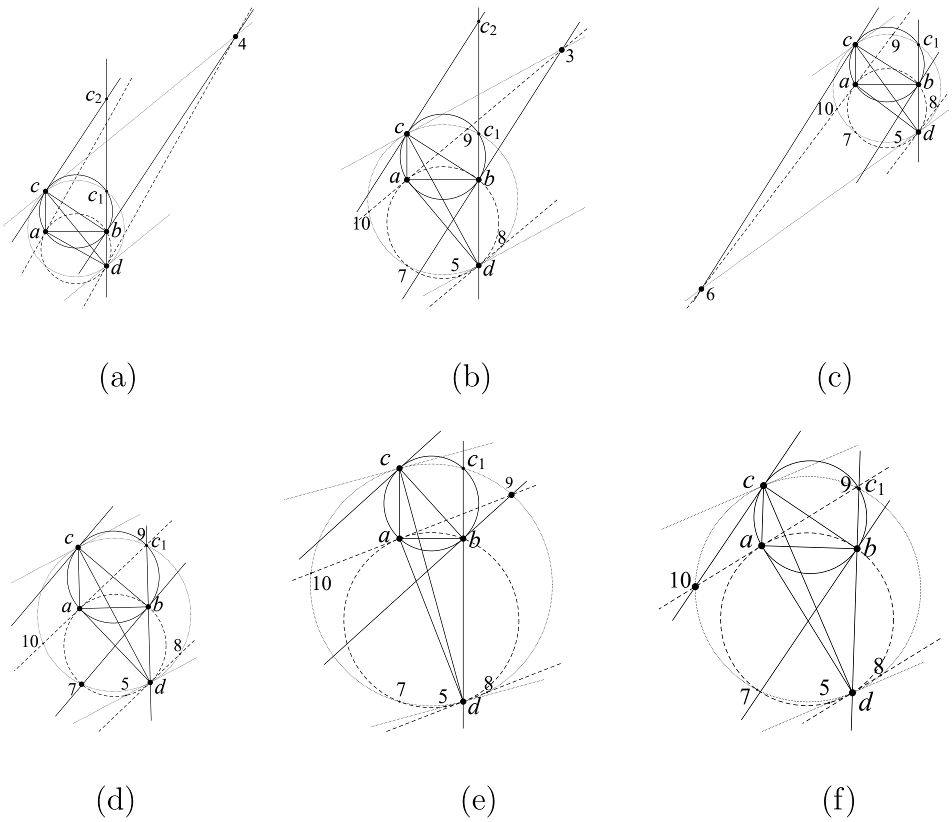


Figure 25: $\angle abc < \frac{\pi}{2}$, $d \in H_{ba}$, $b \in \operatorname{relint} c_1 d$

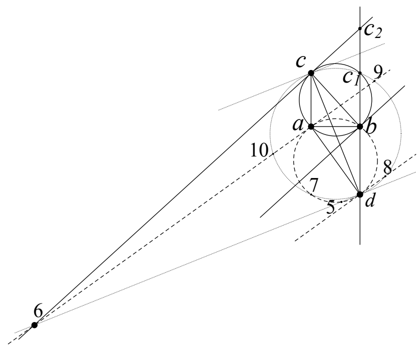


Figure 26: $\angle abc < \frac{\pi}{2}$, $d \in H_{ba}$, $b \in \operatorname{relint} c_1 d$, $|ac| > |bc|$.

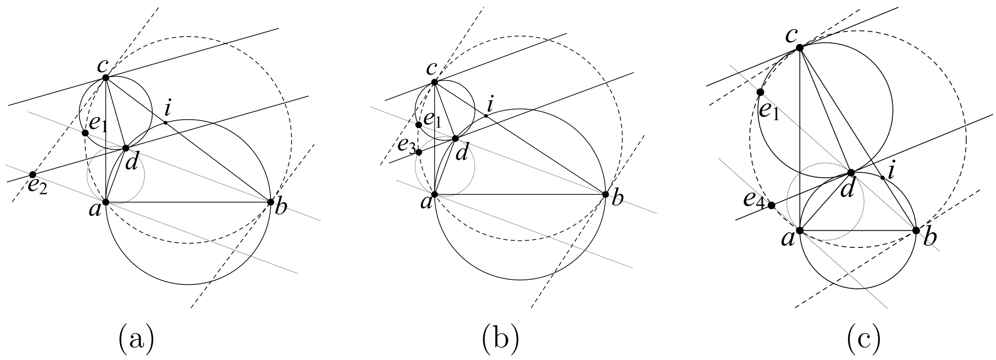


Figure 27: $\angle abc < \frac{\pi}{2}$, $d \in S_{ab}$, $d \in \widehat{ai}$.

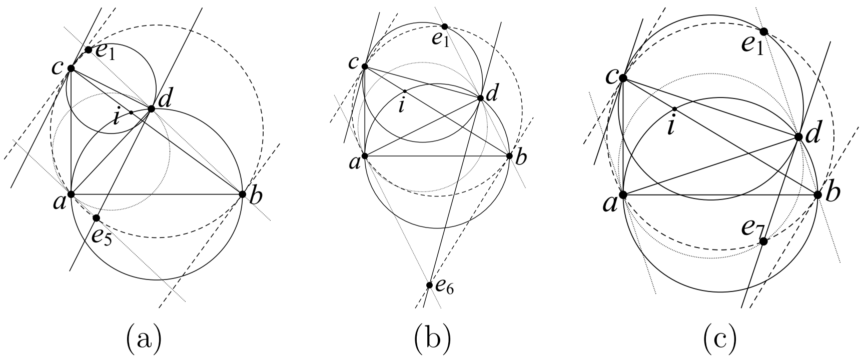


Figure 28: $\angle abc < \frac{\pi}{2}$, $d \in S_{ab}$, $d \in \widehat{ib}$.

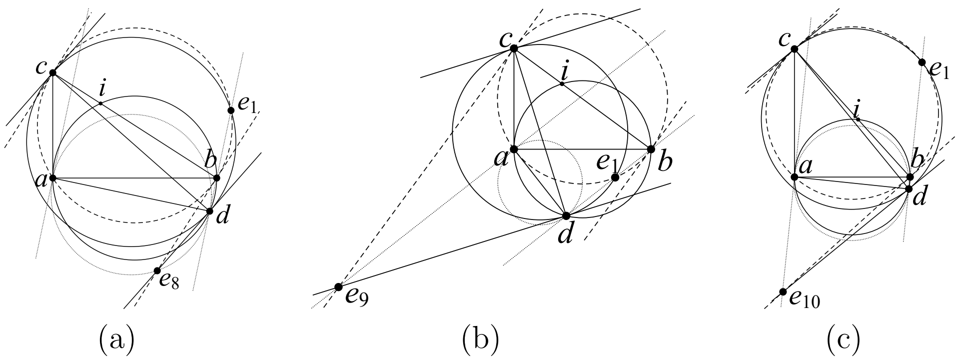


Figure 29: $\angle abc < \frac{\pi}{2}$, $d \in S_{ab}$, $d \in \widehat{ib}$.

actually have $a, b, c \in K_{de}$ satisfying $\{a, b, c\} \in \mathcal{R}$. Thus, we are in Case 1, with d, e instead of a, b and vice-versa.

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