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Gallai's property for graphs in lattices on the torus and the Möbius strip

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Abstract We prove the existence of graphs with empty intersection of their longest paths or cycles as subgraphs of lattices on the torus and the Möbius strip.

Keywords Longest paths · Longest cycles · Square lattice · Hexagonal lattice · Torus · Möbius strip

1 Introduction

We say that a graph is a $\mathbf{P}_k^j$-*graph* ($\mathbf{C}_k^j$-*graph*), if it is a k-connected graph in which any set of j vertices is missed by some longest path (cycle). Such graphs appeared as answers to questions of Gallai [2] and Zamfirescu [10]. Some more questions were asked in [12], and one of them was about the existence of $\mathbf{P}_k^j$-graphs and $\mathbf{C}_k^j$-graphs in (infinite) lattices.

In [8] Nadeem, Shabbir and Zamfirescu proved the existence of $\mathbf{P}_1^1$-, $\mathbf{P}_2^1$- and $\mathbf{C}_2^1$-graphs in the infinite square lattice $\mathcal{L}$ and the infinite hexagonal lattice $\mathcal{H}$ in the plane. Even though the imposed condition is, for $j = 1$, weaker than hypohamiltonicity or hypotraceability, it is not easy to find suitable examples. It is worth noting that hypohamiltonian subgraphs of $\mathcal{L}$ or $\mathcal{H}$ do not exist, since hypohamiltonian graphs cannot be bipartite.

A graph is *hypohamiltonian* (*hypotraceable*) if it has no hamiltonian cycle (path), but after deletion of any vertex it has such a cycle (path). There exists a huge literature on these graphs. See, for example, Chapter 7 in [4]. For more recent advances, see e.g. [6,7].

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In this paper we consider (finite) square and hexagonal lattices on the torus and on the Möbius strip and construct $\mathbf{P}_1^1$-, $\mathbf{P}_2^1$- and $\mathbf{C}_2^1$-subgraphs of these lattices.

Every subgraph of $\mathcal{L}$ or $\mathcal{H}$ is also a subgraph of some lattice on the torus and of some lattice on the Möbius strip, but not conversely. Our examples are interesting not only because they are not realizable in $\mathcal{L}$ or $\mathcal{H}$, but especially because they are also smaller than the corresponding ones obtained in [8].

From the point of view of applications, any such graph is a fault tolerant design. The concept of fault tolerance is associated with reliability, with the absence of breakdowns. It is one of the key criteria in deciding the structures of interconnection networks for parallel and distributed systems (see e.g., [1,3]). Suppose that n processing components are interlinked and l of these units forming a chain or cycle of maximal length are used to solve some task. As a fault tolerant design, any $\mathbf{P}_k^j$- or $\mathbf{C}_k^j$-graph can tolerate the failure of up to j components or communication links, keeping constant performance.

2 Auxilliary results

Let G, H, K, L and M be the graphs homeomorphic to the graphs G' H', K' L' and M' in Fig. 1, 3, 5, 6 and 7, respectively, with the number of vertices of degree 1 or 2 on paths corresponding to edges shown on the respective figures as well.

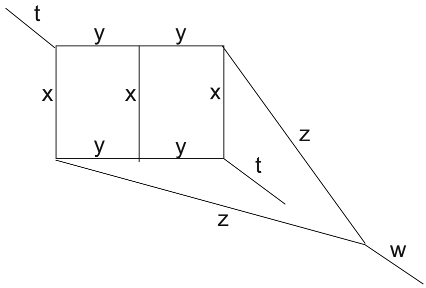

Fig. 1. The graph G'.

Lemma 2.1 *The graph G is a $\mathbf{P}_1^1$-graph, if the following conditions are fulfilled.*

(i) $x \geq 2y + 1$ *and* $x \geq 2z + 1$,
(ii) $t \geq x + z + 1$,
(iii) $w = x + t - z$.

Proof The graph G has the desired property if the paths shown in Fig. 2a–d are of equal lengths and the paths of Fig. 2e–h are not longer. The lengths a, b of the paths in Fig. 2a, b are

$$a = 3x + 2y + 2t + 5,$$
$$b = 2x + 2y + z + t + w + 5,$$

and the lengths of the paths in Fig. 2e–h are

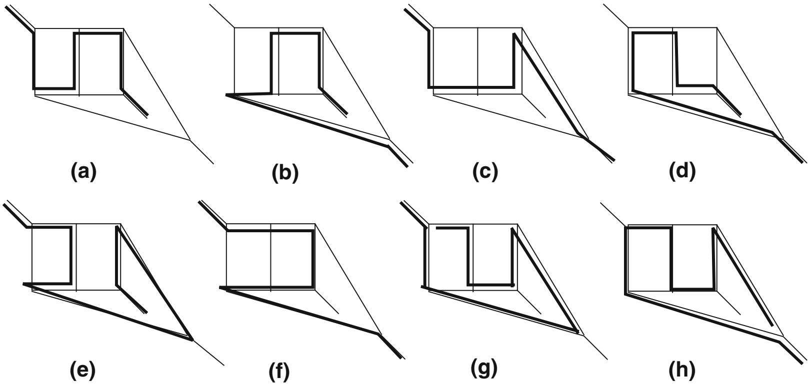

Fig. 2.

$$
\begin{aligned}
e &= 2x + 2y + 2z + 2t + 6,\\
f &= x + 4y + z + t + w + 6,\\
g &= 3x + 2y + 2z + t + 6,\\
h &= 3x + 2y + 2z + w + 6,
\end{aligned}
$$

respectively. The paths of Fig. 2b–d obviously have equal lengths. The conditions in the statement imply $a = b$ and $a \geq \max\{e, f, g, h\}$ indeed. □

The graph G corresponding to the smallest solution $x = 1$, $y = z = 0$, $t = 2$, $w = 3$ was discovered by Schmitz [9].

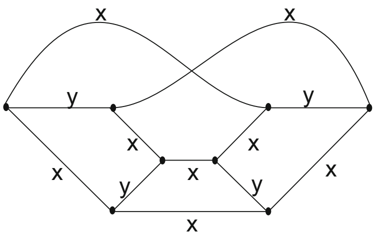

Fig. 3. The graph H'.

Lemma 2.2 *The graph H is a $\mathbf{C}_2^1$-graph, if $y \geq 2x + 1$.*

Proof To prove this lemma, we use the same technique as for Lemma 2.1. Here the longest cycles of H have empty intersection if the cycle of Fig. 4a is among the longest cycles and the cycles of Fig. 4b, c are not longer. Indeed, the lengths

$$
\begin{aligned}
a &= 4x + 3y + 7,\\
b &= 8x + 8,\\
c &= 6x + 2y + 8,
\end{aligned}
$$

of these cycles satisfy $a \geq \max\{b, c\}$, because this is equivalent to $y \geq 2x + 1$. □

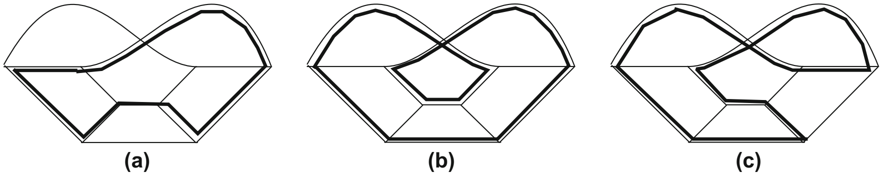

Fig. 4.

The graph H obtained for $x = 0$, $y = 1$ was first presented by the second author (see [5]). The following lemmas are taken from [8]. See Figs. 5–7.

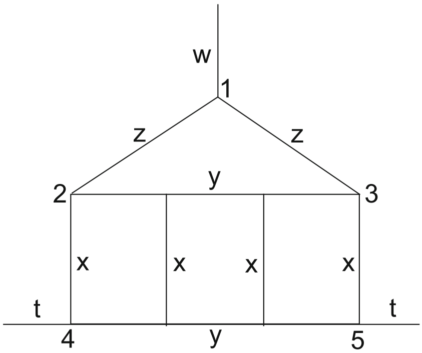

Fig. 5. The graph K'.

Lemma 2.3 *The longest paths of K have empty intersection if the following conditions are verified.*

(i) $2x \geq y + 2z + 1$,
(ii) $t \geq y + 2z + 1$,
(iii) $t \geq x + z + 1$,
(iv) $w = x + t - z$.

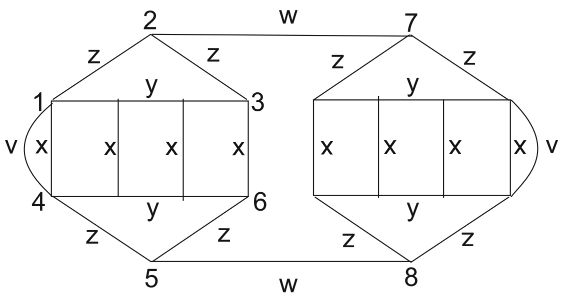

Fig. 6. The graph L'.

Lemma 2.4 *Let $x \geq v$. The longest paths of L have empty intersection if the following conditions are fulfilled.*

(i) $v \geq y + 2z + 1$,
(ii) $x + v = y + 2z + w + 1$.

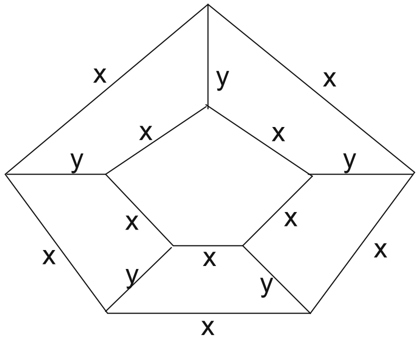

Fig. 7. The graph M'.

Lemma 2.5 *The longest cycles of M have empty intersection if* $2y \geq 3x + 1$.

3 Embeddings in toroidal lattices

Consider an $(m+1) \times (n+1)$ rectangle (with $(m+1)(n+1)$ vertices) in $\mathcal{L}$. By identifying opposite vertices on the boundary as indicated on Fig. 8a, we obtain the *toroidal square lattice* $\mathcal{L}^T_{m,n}$. It has mn vertices. Similarly, we define the *toroidal hexagonal lattice* $\mathcal{H}^T_{m,n}$ for even values of m and n and with mn vertices (see Fig. 8b).

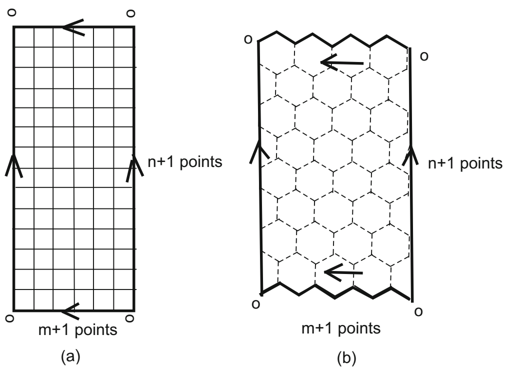

Fig. 8.

In this section we look for $\mathbf{P}^1_1$-, $\mathbf{P}^1_2$- and $\mathbf{C}^1_2$-subgraphs of these toroidal lattices.

A graph is called *planar* if it admits an embedding into the Euclidean plane.

Theorem 3.1 *There exists a spanning planar* $\mathbf{P}^1_1$*-subgraph of* $\mathcal{L}^T_{5,4}$.

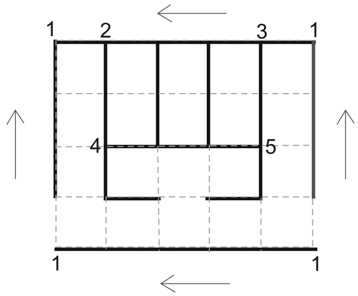

Fig. 9.

Proof For $y = z = 0$, $x = 1$, $t = 2$ and $w = 3$, the conditions of Lemma 2.3 are verified and the corresponding graph K of order 20 has the desired property. It is obviously planar. Figure 9 reveals an embedding of K in $\mathcal{L}^T_{5,4}$ as a spanning subgraph. □

Theorem 3.2 *There exists a planar* $\mathbf{P}^1_2$*-graph in* $\mathcal{L}^T_{10,10}$ *of order* 80.

Proof Take $y = z = 1$, $x = v = w = 4$ in Lemma 2.4. It can be easily checked that the chosen values y, z, x, v and w satisfy its conditions and the resulting graph L is a planar $\mathbf{P}^1_2$-graph of order 80. Figure 10 presents an embedding of L in the toroidal lattice $\mathcal{L}^T_{10,10}$. □

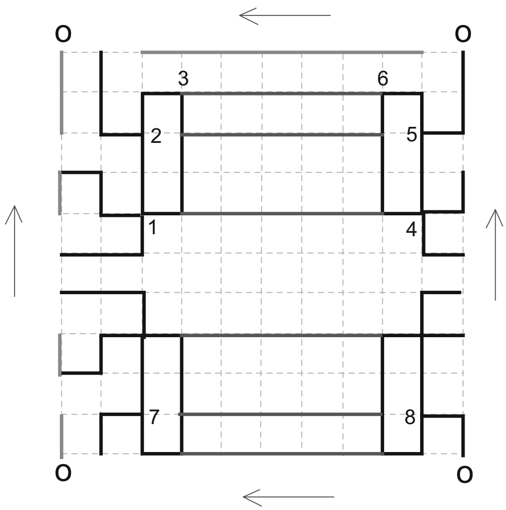

Fig. 10.

Theorem 3.3 *In* $\mathcal{L}^T_{5,3}$ *we have a spanning planar* $\mathbf{C}^1_2$*-graph.*

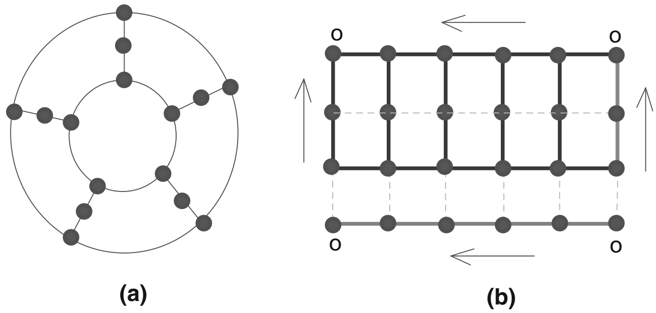

Fig. 11.

Proof For $x = 0$ and $y = 1$ in Lemma 2.5, the resulting graph M becomes Thomassen's graph of order 15 (see Fig. 11a and [11]), which is the smallest known $\mathbf{C}_2^1$-graph in the plane. This graph is embeddable in $\mathcal{L}_{5,3}^T$, see Fig. 11b. □

Theorem 3.4 *There exists a planar* $\mathbf{P}_1^1$*-subgraph of* $\mathcal{H}_{12,6}^T$ *of order* 58.

Proof The conditions of Lemma 2.3 are also verified if we take $y = 1$, $z = 3$, $x = 4$, $t = 8$ and $w = 9$, and the corresponding graph K is of order 58. Figure 12 reveals an embedding of K in $\mathcal{H}_{12,6}^T$. □

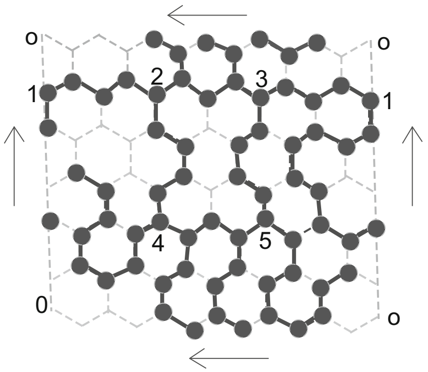

Fig. 12.

Theorem 3.5 *The lattice* $\mathcal{H}_{10,4}^T$ *contains a planar* $\mathbf{C}_2^1$*-subgraph of order* 30.

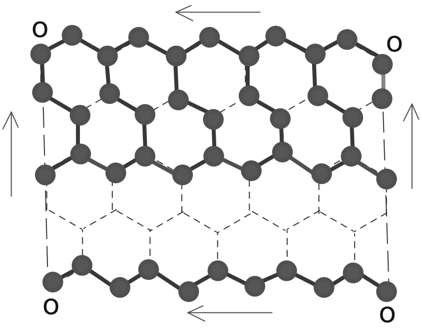

Fig. 13.

Proof To obtain a graph as required we will use Lemma 2.5 once again. Take $x = 1$ and $y = 2$, which satisfy the conditions of the lemma, and we are led to a graph M of order 30, an embedding of which in $\mathcal{H}^T_{10,4}$ is shown in Fig. 13. □

We conjecture the orders 20, 80, 15, 58 and 30 of the graphs presented in Theorems 3.1–3.5 and the orders 20, 100, 15, 72 and 40 of the respective toroidal lattices to be minimal.

4 Embeddings in lattices on Möbius strips

To obtain the lattice graph $\mathcal{L}^M_{m,n}$ on the *Möbius strip*, we identify opposite vertices taken in reverse order on parts of the boundary of an $(m+1) \times n$ rectangle, as indicated in Fig. 14a. Similarly, according to Fig. 14b, we define $\mathcal{H}^M_{m,n}$.

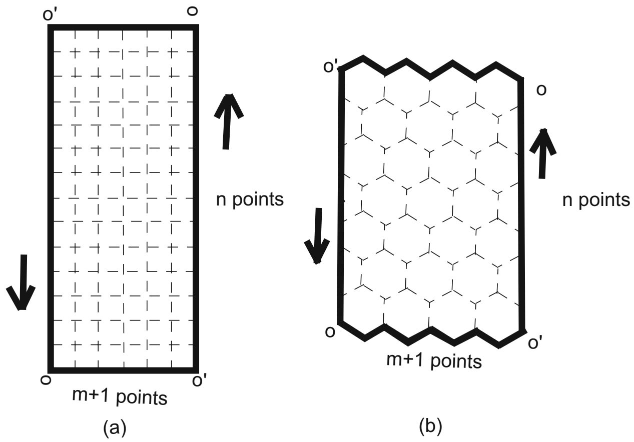

Fig. 14.

Our next theorems are about the existence of $\mathbf{P}^1_1$-, $\mathbf{P}^1_2$- and $\mathbf{C}^1_2$-subgraphs of $\mathcal{L}^M_{m,n}$ and $\mathcal{H}^M_{m,n}$.

Theorem 4.1 *In $\mathcal{L}^M_{3,8}$ there is a planar $\mathbf{P}^1_1$-subgraph of order* 17.

Proof In the plane the smallest known $\mathbf{P}^1_1$-subgraph is Schmitz's graph [9] shown in Fig. 15a, whose order is 17. We succeed to embed this graph in $\mathcal{L}^M_{3,8}$ as shown in Fig. 15b. □

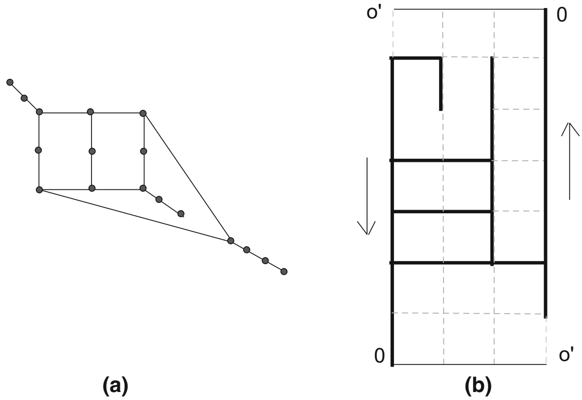

Fig. 15.

Theorem 4.2 *The lattice* $\mathcal{L}^M_{10,14}$ *contains a planar* $\mathbf{P}^1_2$*-graph of order* 112.

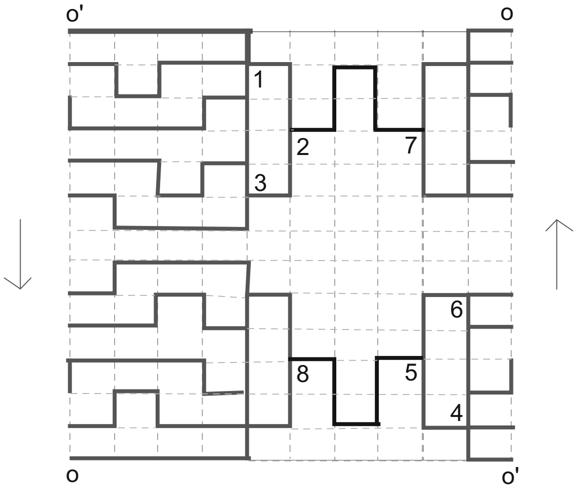

Fig. 16.

Proof Let us take $y = 1$, $z = 2$, $x = v = w = 6$ in Lemma 2.4. The conditions of the lemma are verified for these values and the resulting graph K is a planar $\mathbf{P}^1_2$-graph of order 112. Figure 16 is an embedding of K in $\mathcal{L}^M_{10,14}$. □

Theorem 4.3 *There exists a spanning* $\mathbf{C}^1_2$*-subgraph of* $\mathcal{L}^M_{4,3}$.

Proof It is clear that the graph H with $x = 0$ and $y = 1$ satisfies the conditions of Lemma 2.2, and the corresponding graph H is a $\mathbf{C}^1_2$-graph of order 12. Figure 17 presents an embedding of H as a spanning subgraph of $\mathcal{L}^M_{4,3}$. □

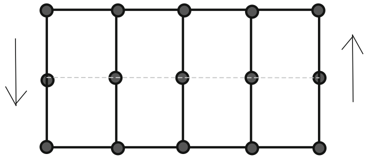

Fig. 17.

Theorem 4.4 *The lattice* $\mathcal{H}^M_{7,9}$ *contains a planar* $\mathbf{P}^1_1$*-graph of order* 46.

Proof To prove the existence of a $\mathbf{P}^1_1$-graph in $\mathcal{H}^M_{7,9}$, we take $x = 4$, $y = z = 1$, $t = 6$ and $w = 9$ in Lemma 2.1. The resulting graph G is of order 46. Moreover, G is embeddable in $\mathcal{H}^M_{7,9}$. In Fig. 18 we show an embedding of G. □

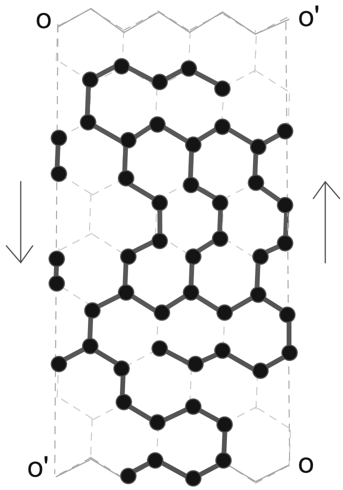

Fig. 18.

Theorem 4.5 *There exists a* $\mathbf{C}^1_2$*-subgraph of* $\mathcal{H}^M_{8,4}$ *of order* 32.

Proof The conditions of Lemma 2.2 are also satisfied for $x = 1$ and $y = 4$ and the resulting graph H is of order 32. A spanning subgraph of $\mathcal{H}^M_{8,4}$ isomorphic to H is shown in Fig. 19. □

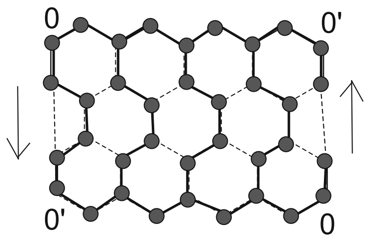

Fig. 19.

We conclude this section conjecturing that in Theorems 4.1–4.5 too, the orders of the lattices and those of the embedded graphs are minimal.

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