Fixed Point Theory, 18(2017), No. 1, 407-412
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

FIXED CHORDS AND DISCS IN CONVEX BODIES

TUDOR ZAMFIRESCU

*Fachbereich Mathematik, Technische Universität Dortmund
44221 Dortmund, Germany
and
Simion Stoilow Institute of Mathematics, Romanian Academy
Bucharest, Roumania
and
College of Mathematics and Information Science,
Hebei Normal University,
050024 Shijiazhuang, P.R. China
E-mail: tudor.zamfirescu@mathematik.uni-dortmund.de

Abstract. We prove here that, in the sense of Baire categories, most convex bodies possess fixed chords and fixed discs.
Key Words and Phrases: fixed chords, fixed discs, convex bodies, Baire categories.
2010 Mathematics Subject Classification: 52A20, 52A15, 54E52, 54H25, 47H10.

1. Introduction

Baire categories have been successfully used, especially in Analysis. They penetrated Convexity when Klee published his pioneering work in 1959 [2]. Since then several sometimes surprising results have been obtained about most convex bodies, i.e. all of them except for a set of first Baire category; see the book [6] and the surveys [1] and [4]. Brouwer and Schauder's fixed point theorems have also been generically treated [5]. However, applications in fixed point theory did not abound.

In this paper we present Baire category results belonging to both the theory of convex bodies and that of fixed points. In fact, there will be 1- or 2-dimensional balls which will be fixed. Let us become more explicit.

Let X be a connected metric space and $\mathcal{C}$ a space of continua in X. A continuous function $f : [0, 1] \to \mathcal{C}$ with all sets $f(t)$ pairwise isometric ($t \in [0, 1]$) is a *movement* of $f(0)$. We say that $C \in \mathcal{C}$ is *fixed* if the only possible movement of C is the constant function $f(t) = f(0) = C$ for all t. As a referee remarked, finding the fixed continua in $\mathcal{C}$ means finding the strict fixed points of the multivalued mapping $F : \mathcal{C} \to P(\mathcal{C})$ defined by $F(C) = \{f(t) : t \in [0, 1], f \text{ movement of } C\}$.

Let K be a convex body in $\mathbb{R}^d$. A *chord* of K is a line-segment xy with $x, y \in K$. We take X to be K and $\mathcal{C}$ the space of all chords of K. Does K possess fixed chords?

Some examples show different answers. While an ellipsoid with different axis lengths has a unique fixed chord, the ball has no fixed chord, and the regular n-gon has n fixed chords for odd n.

Is the ball a great exception (not possessing any fixed chords)? No, we immediately see that all convex bodies with constant width enjoy the same lack of fixed chords! And ellipsoids with two equally long axes and a shorter third one are further examples.

How do most convex bodies behave? Have they any fixed chords? If yes, how many? We shall show here that most convex bodies in the space $\mathcal{K}$ of all convex bodies in $\mathbb{R}^d$ have infinitely many fixed chords, no pair of which meet or have same length.

2. Very many continua with fixed subcontinua

Let us start with a rather general case. We consider the space $\mathcal{D}$ of all continua in $\mathbb{R}^d$, and take $X \in \mathcal{D}$.

Theorem 2.1. *In most continua of $\mathbb{R}^d$, every subcontinuum is fixed.*

Proof. Let $\mathcal{D}_n$ be the set of continua in $\mathbb{R}^d$ possessing a subcontinuum of diameter at least $1/n$ which is not fixed. We prove that $\mathcal{D}_n$ is nowhere dense in $\mathcal{D}$, and this will establish the theorem.

Obviously, every continuum in $\mathcal{D}$ can be approximated by a connected geometric graph, i.e. a connected finite union F of line-segments. This can be done such that F includes no line-segment of length $1/n$. Then, clearly, $F \notin \mathcal{D}_n$. The set of all connected geometric graphs including no line-segment of length $1/n$ is open. Thus, $\mathcal{D}_n$ is nowhere dense.

There are, however, many important continua in $\mathbb{R}^d$ with lots of non-fixed subcontinua. So, for example, take a convex body $K \subset \mathbb{R}^d$. In K, all compact convex subsets are non-fixed, if we disregard a nowhere dense family of exceptions. Nevertheless, restricting the set of considered compact convex subsets yields more interesting results.

3. Many convex bodies without fixed chords

We prove here the following theorem.

Theorem 3.1. *The set of convex bodies without any fixed chords is dense in the space $\mathcal{K}$ of all convex bodies.*

Proof. Let $\mathcal{O} \subset K$ be open. Choose a polytope $P \in \mathcal{O}$ with no pair of parallel faces (of positive dimension). There exist fixed chords in P: at least the chords of maximal length will be fixed. Let $\mathcal{F}$ be the (finite) family of all fixed chords of P. For every vertex v of P choose a neighbourhood $N_v \subset \mathbb{R}^d$ of v such that $N_v \cap N_u = \emptyset$ for $v \neq u$, and for arbitrary non-empty subsets $N'_v \subset N_v$, $\operatorname{conv} \bigcup_v N'_v \in \mathcal{O}$. Now choose $c = vw \in \mathcal{F}$. The angle between vw and any edge vu of P must be acute. Let $\varepsilon > 0$. For ε small enough,

$$\{v^* \in P : \|v^* - w\| > \|v - w\| - \varepsilon\}$$

has a component $V_c \subset N_v$. Put $P_c = P \setminus V_c$.

Now let $c' \in \mathcal{F}$ be different from c. If c and c' have no common endpoint or $c \cap c' = \{w\}$, we proceed with c' as with c. If $c' = vw'$, then P_c has a longest chord $v'w'$ among all chords joining w' to points in $Z = \overline{V_c} \cap P_c$, where $\overline{V_c}$ denotes the closure of V_c. Put

$$m = \max_{z \in Z} \|z - w'\|,$$
$$n = \min_{z \in Z} \|z - w'\|.$$

Thus, $\|v' - w'\| = m$. Let $n < s < m$. Then

$$\{v^* \in P_c : \|v^* - w'\| > s\}$$

has a component $V_{c'} \subset N_v$, such that the boundary bdV_c of V_c is not included in $\overline{V_{c'}}$. Consider the convex body $P_c \setminus V_{c'}$. Its boundary contains two spherical regions, both in N_v, with w and w' as centres.

Repeat this procedure for all chords in $\mathcal{F}$. We eventually obtain a convex body in $\mathcal{O}$ without fixed chords.

4. More convex bodies with many fixed chords

We shall see in this section that, in the sense of Baire categories, most convex bodies do have fixed chords, they even have infinitely many of them.

A chord Δ of $K \in \mathcal{K}$ will be called *long* if for some set $V \supset \Delta$ open in $\mathbb{R}^d$, K has no chord included in V and longer than Δ. Let $\mathcal{L}_K$ be the set of all long chords of K.

Lemma 4.1. *For any $K \in \mathcal{K}$, all chords of a component of $\mathcal{L}_K$ have same length.*

We leave the elementary proof to the reader.

Lemma 4.2. *For most convex bodies, $\mathcal{L}_K$ has infinitely many components.*

Proof. Let $\mathcal{K}_n$ be the set of all convex bodies K for which $\mathcal{L}_K$ has at most n components. We show that $\mathcal{K}_n$ is nowhere dense.

Let $\mathcal{O} \subset K$ be open and choose a polytope $P \in \mathcal{O}$ as in the preceding proof. In P, the (finite) set $\mathcal{F}$ of fixed chords coincides with the set of long chords. Take $vw \in \mathcal{F}$. In a plane orthogonal to vw, consider a convex $(n+1)$-gon Q with v in its relative interior. In the polytope $P' = \text{conv}(P \cup Q)$, each chord vu is long if u is a vertex of Q. If Q and the neighbourhood $\mathcal{V}$ of P' are small enough, then $\mathcal{V} \subset O$ and for any convex body $K \in \mathcal{V}$, close to each chord vu there is a long chord, and the $n+1$ chords obtained this way are in pairwise different components. This holds due to Lemma 1 and because any chord outside a small neighborhood of vu is strictly shorter than vu. Thus, $\mathcal{V} \cap \mathcal{K}_n = \emptyset$ and $\mathcal{K}_n$ is nowhere dense indeed.

Hence $\bigcup_n \mathcal{K}_n$ is of first category, and the lemma is proven.

Lemma 4.3. *For most convex bodies, $\mathcal{L}_K$ is totally disconnected.*

Proof. Let $\mathcal{K}_n$ be the set of all convex bodies possessing a long chord whose component has diameter at least $1/n$. Consider the same polytope P as in the proofs of Theorem 2 or Lemma 2. If $\varepsilon > 0$ is small enough, for no convex body $K \subset P + \varepsilon B$, $\mathcal{L}_K$ has a component of diameter at least $1/n$. (Here, B is the unit ball centred at $\mathbf{0}$.) This yields that $\mathcal{K}_n$ is nowhere dense, whence the lemma.

Theorem 4.4. *Most convex bodies have infinitely many fixed chords.*

Proof. Since every long chord is fixed if its component contains no other chord, it suffices to combine Lemma 2 and Lemma 3.

It is interesting to note that isolated long chords and fixed chords are not equivalent notions, see the Figure 1.

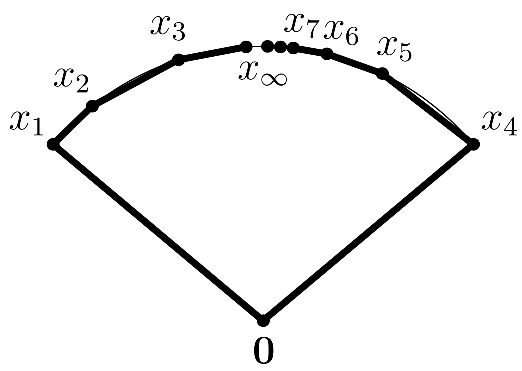

Figure 1. On the circular arc, $x_n \to x_\infty$. In $K = \text{conv}\{\mathbf{0}, x_\infty, x_1, x_2, ...\}$, the long chord $\mathbf{0}x_\infty$ is fixed, but not isolated.

5. Fixed discs

In order to better visualize the matter and not unnecessarily complicate the proofs with technicalities derived from arbitrary dimension, let $d = 3$.

We now consider a 2-dimensional disc included in the convex body $K \in \mathcal{K}$ instead of a chord, and define *fixed discs* analogously.

How many contact points has a fixed disc with the boundary of K ?

It must have at least two contact points, because otherwise it can be moved in its plane. Interestingly, if a disc in K has exactly one contact point (in bd$K$), it may be impossible to move it thereby keeping fixed the contact point!

Consider the intersection (with non-empty interior) L of two congruent distinct balls B_1, B_2. The disc $D = \text{conv}(\text{bd}B_1 \cap \text{bd}B_2)$ is obviously fixed in L, and the set $D \cap \text{bd}L$ is infinite.

If $\mathbf{0}$ is the centre of D and the z-axis is orthogonal to aff D, then

$$L' = \{(x/2, y, z) : (x, y, z) \in L\}$$

has a single fixed disc, namely $D/2$, which has precisely two contact points with L'.

Suppose that K is a smooth convex body and the plane Π cuts K along a planar convex body K_Π. Further, assume that K_Π has an inscribed circular disc having 4 contact points with the relative boundary relbd K_Π of K_Π. If no tangent plane at each of the 4 points is orthogonal to Π and the acute angles between them and Π are formed alternately toward the two sides of Π as the points are taken in their circular order, then we say that the tangent planes of bd K at the 4 points form *alternately acute* angles with Π.

Lemma 5.1. *If $K \in \mathcal{K}$ is smooth, the disc $D \subset K$ has* 2 *or* 3 *contact points with* bd$K$ *and the tangent planes at those points to* bd$K$ *are not orthogonal to* aff$D$*, then D is not fixed.*

We leave the easy proof to the reader.

Theorem 5.2. *Most convex bodies K admit a fixed disc, which has exactly* 4 *contact points in* bd$K$.

Proof. First of all, we work in the space $\mathcal{K}'$ of all smooth convex bodies, which is residual in $\mathcal{K}$ by Klee's theorem [2].

Let $\mathcal{K}_n$ be the space of all convex bodies in $\mathcal{K}'$ admitting a disc which cannot be moved at distance $1/n$. We show that, for every n, the complement of $\mathcal{K}_n$ in $\mathcal{K}'$ is nowhere dense. Thus, the complement of $\bigcap_{n=1}^{\infty} \mathcal{K}_n$ is of first category, and most convex bodies belong to all $\mathcal{K}_n$.

Take $\mathcal{O} \subset \mathcal{K}'$ open, and $K \in \mathcal{O}$. Let $D \subset K$ be a disc of maximal radius. Then D is inscribed in $K' = K \cap \text{aff}D$, where aff$D$ means the plane determined by D. Therefore, D has at least two contact points on relbd K'. Let C_K be the set of these contact points. If card C_K is 2 or 3, then the tangent planes at (some or all of) those points must be orthogonal to aff$D$, by Lemma 4, and put $C' = C_K$. If in C_K there are at least 4 contact points, choose 4 among them having the centre c of D in their convex hull, and define C' to be this 4-point set. Now, take a 4-point set C'' approximating C', such that its convex hull contains c in its interior. If C' has less than 4 points, choose the additional points in C'' close to those contact points of C' where the tangent planes of bd K are orthogonal to aff$D$. Take a convex body K' in $\mathcal{K}'$ close to K with $C_{K'} = C''$ and having tangent planes alternately acute with aff$D$ at the four contact points. Then D is fixed in K' and any convex body close enough to K' belongs to $\mathcal{K}_n$, whence the complement of $\mathcal{K}_n$ is nowhere dense.

We have proved above not only that most K have a fixed disc D, but also that $D \cap \text{bd}K$ contains at least 4 points for these K.

To prove that, for most K, card $(D \cap \text{bd}K) = 4$, define $\mathcal{K}_n$ as set of those $K \in \mathcal{K}$ which, besides admitting a disc which cannot be moved at distance $1/n$, also have no 5 points among its contact points with bd$K$ at mutual distances at least $1/n$. The proof above provides now card $(D \cap \text{bd}K) = 4$ for most $K \in \mathcal{K}$.

Notice a certain similarity between this proof and the argument in [3].

Acknowledgement. The author thanks a referee for his suggestions and for pointing out some errors in the manuscript. Also, the author acknowledges partial support via a grant of the Roumanian National Authority for Scientific Research, CNCS - UEFISCDI, project number PN-II-ID-PCE-2011-3-0533.

References

[1] P. Gruber, *Baire categories in convexity*, Handbook of Convex Geometry, (P. Gruber and J. Wills -Eds.), Elsevier Science, Amsterdam, 1993.

[2] V. Klee, *Some new results on smoothness and rotundity in normed linear spaces*, Math. Ann., **139**(1959), 51-63.

[3] T. Zamfirescu, *Inscribed and circumscribed circles to convex curves*, Proc. Amer. Math. Soc., **80**(1980), 455-457.
[4] T. Zamfirescu, *Baire Categories in Convexity*, Atti Sem. Mat. Fis. Univ. Modena, **39**(1991), 139-164.
[5] T. Zamfirescu, *A generic view on the theorems of Brouwer and Schauder*, Math. Z., **213**(1993), 387-392.
[6] T. Zamfirescu; *The Majority in Convexity*, Ed. Universităţii Bucureşti, 1st Edition, 2009.

Received: June 24, 2014; Accepted: March 3, 2015.